Problems for **Einstein's Gravity: Black Holes to Gravity Waves**
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Lecture 1

Probem 1: Velocity Addition Rule. Given the Lorentz transformation describing the relation between two inertial frames moving with a relative velocity v along the x-axis, verify the following rule for the transformation of velocities between the two frames:

$$\begin{aligned} V^{x'} &= \frac{V^x - v}{1 - \frac{vV^x}{c^2}} \\ V^{y'} &= \frac{V^y}{1 - \frac{vV^x}{c^2}} \sqrt{1 - \frac{v^2}{c^2}} \\ V^{z'} &= \frac{V^z}{1 - \frac{vV^x}{c^2}} \sqrt{1 - \frac{v^2}{c^2}} \end{aligned}$$

where $V^x = \frac{dx}{dt}$, $V^{x'} = \frac{dx'}{dt'}$, etcetera.

Show that in general if $|\vec{V}| = c$ then $|\vec{V}'| = c$. Also show that accelerations are absolute, *i.e.,* if $\left|\frac{d^2\vec{x}}{dt^2}\right| \neq 0$ in one frame, then the same is true in all inertial frames.

Probem 2: Doppler shift. An observer O at rest emits a phone along the x-axis with wave (four-)vector:

$$\mathbf{k} = (k^t, k^x, k^y, k^z) = (\omega/c, \omega/c, 0, 0)\,.$$

What is the photon frequency measured by O? The photon is seen by an observer O' moving with speed v towards negative x. What is the photon frequency measured by O'?

Generalize this result for an observer O'' moving with a general velocity $\vec{v}$, *i.e.,* $\vec{v}$ is not necessarily parallel to the x-axis.

Problem 3: Accelerating Observer. Consider an observer O moving along the following world-line:

$$t(\sigma) = \frac{1}{a}\,\sinh\sigma\,, \qquad x(\sigma) = \frac{1}{a}\,\cosh\sigma\,.$$

Sketch the world-line in Minkowski space. What is the proper time τ along the trajectory? (Assume the observer's clock is set so that $\tau = 0$ at $t = 0$.) What is the observer's four-velocity? Show that the three-velocity approaches the speed of light as $\sigma \to \pm\infty$.

Evaluate the observer's four-acceleration, defined as $a^\alpha = \frac{du^\alpha}{d\tau}$. (This is the quantity that appears in the relativistic version of Newton's second law, *i.e.,* $m\,\mathbf{a} = \mathbf{F}$ where $\mathbf{F}$ is the four-force.) Show that $\mathbf{u} \cdot \mathbf{a} = 0$ in the above example. Show that $\mathbf{u} \cdot \mathbf{a} = 0$ in general.

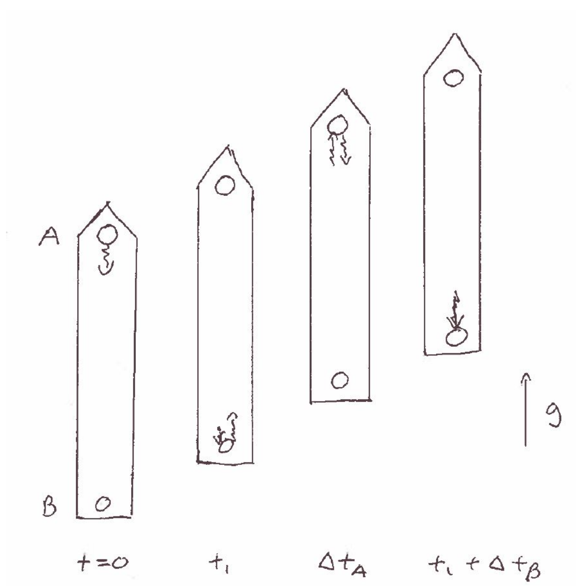
Imagine that an observer O' sits at rest at $x = 0$ and emits a constant stream of photons with $\mathbf{k}$ as in the previous problem. Evaluate the frequency seen by O as a function of her proper time. When is $\omega(\tau) = \omega$?

Lecture 2

There are too many problems for this lecture for you to do in one day. So I would suggest that you focus on problem 1 and work on others that you find interesting today. You can come back to the other problems later.

Probem 1: To understand that gravity effects the way clocks run, we will consider the problem of a rocket ship that is accelerating upward at a constant rate. The equivalence principle says that the physics observed in this rocket ship should be the same as in a rocket ship sitting at rest in a uniform gravitational field. Hence let us begin by considering an experiment of sending light signals up and down in the accelerating rocket and then infer the result for the same experiment in the rocket ship when it is in a uniform gravitational field.

The rocket ship has a height h.[1] The rocket begins at rest, *i.e.,* $\vec{v} = 0$, and then accelerates upward with acceleration g. A pulse of light is emitted from the top of the rocket down towards the bottom and is reflected back and forth between mirrors at the top and bottom of the rocket, as shown:



Using simple classic formulae for the motion of the rocket, calculate Δt_A, the time between the

[1] All of the velocities are assumed to be much smaller than c in this problem and so the effects of special relativity, *e.g.,* length contraction, on the rocket ship are not important. Implicitly, this will also imply that $gh/c^2 \ll 1$, which, in the gravitational problem, means that the change in the gravitational potential energy over the length of the ship is much less that the rest mass energy of any particle.

departure and arrival of the signal from the top of the ship, and Δt_A, the same for the bottom of the ship. Verify the following result:

$$\Delta t_A = \left(1 + gh/c^2\right) \Delta t_B \,.$$

Now the equivalence principle says that the same effect holds for the rocket sitting at rest in a uniform gravitational field (where the acceleration due to gravity in the lab is g downward). Hence the time intervals measured by clocks at the top and bottom are different with

$$\Delta\tau(\text{height } h) = \left(1 + gh/c^2\right) \Delta\tau(\text{height } 0) \,.$$

I am using τ here to emphasize that these are proper time intervals measured locally by (stationary) clocks at different locations. In a uniform gravitational field, we can think of gh as the change in the gravitational potential, *i.e.,*

$$gh = \Delta\Phi = \Phi(\vec{x}_{\text{top}}) - \Phi(\vec{x}_{\text{bottom}}) \,. \tag{0.1}$$

So in comparing clock rates throughout a gravitational field, we can write

$$\Delta\tau(\vec{x}) = \left(1 + \Phi(\vec{x})/c^2\right) \, \Delta\tau(\text{at } \Phi = 0) \,.$$

This result then tells us that in the presence of a gravitational potential, we need to fix up our distance rule. The new rule is then given by the 'clock' metric

$$ds^2 = -\left(c^2 + 2\Phi(\vec{x})\right) \, dt^2 + dx^2 + dy^2 + dz^2 \,.$$

This is the first gravitational correction to the metric which is a good approximation in the regime where $\Phi(\vec{x})/c^2 \ll 1$ — see the footnote. Note that the coordinate t is a label that we use everywhere in the spacetime but we can think of it as the proper time measured by stationary clocks at the location where $\Phi(\vec{x}) = 0$, *e.g.,* for a local mass distribution, we usually set the gravitational potential to zero at infinity.

Problem 2: Consider the transformation from Cartesian coordinates $\{x, y, z\}$ to polar coordinates $\{r, \theta, \phi\}$:

$$x = r \sin\theta \cos\phi \,, \quad y = r \sin\theta \sin\phi \,, \quad x = r \cos\theta \,.$$

Verify that the standard line element for infinitesimal displacements in Minkowski space transforms to

$$ds^2 = -dt^2 + dr^2 + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right) \,.$$

Write out the inverse metric $\eta^{\alpha\beta}$ in these coordinates.

Problem 3: Relativity and GPS. The Global Positioning System (GPS) has become a part of daily life. The nominal GPS configuration consists of a network of 24 satellites in high orbits around the Earth. Each satellite in the GPS constellation orbits at an altitude of about 20,000 km from the ground, and has an orbital speed of about 14,000 km/hour (the orbital period is roughly 12 hours). The satellite orbits are distributed so that at least 4 satellites are always visible from any point on the Earth at any given instant. Each satellite carries with it an atomic clock that "ticks" with a nominal accuracy of 1 nanosecond (1 billionth of a second). A GPS receiver in an airplane, a car or your cellphone determines its current position and course by comparing the time signals it receives from the currently visible GPS satellites and triangulating on the known positions of each satellite. The precision achieved is remarkable: even a simple GPS receiver can determine your absolute position on the surface of the Earth to within 5 to 10 meters in only a few seconds. To achieve this level of precision, the clock ticks from the GPS satellites must be known to an accuracy of 20-30 nanoseconds.

However, because the satellites are constantly moving relative to observers on the Earth, effects predicted by Special Relativity and General Relativity must be taken into account to achieve the desired accuracy. Because an observer on the ground sees the satellites in motion relative to them, Special Relativity predicts that we should see their clocks ticking more slowly. Roughly how much will the atomic clocks on the satellites fall behind clocks on the ground in a day because of this effect? On the other hand, the satellites are in orbits high above the Earth, and so General Relativity predicts that the clocks on the satellites appear to be ticking faster than identical clocks on the ground. Roughly how much will the satellite clocks get ahead of ground-based clocks in a day because of this effect? (You may have to look up some of the parameters on Google for these calculations.)

An interesting discussion of this topic appears in: *Relativity and the Global Positioning System*, Neil Ashby, Physics Today, May 2002, 41.

Problem 4: Rindler space. Consider the two-dimensional geometry described by following line element:

$$ds^2 = -X^2\, dT^2 + dX^2\,.$$

Evaluate all of the nonvanishing components of the Christoffel symbol for this metric.

Solve for general null geodesics travelling through this geometry. Sketch a diagram of these geodesics. (Optional: You may also try solving for time-like geodesics but technically, this is a little more challenging.)

Consider the coordinate transformation:

$$t = X\, \sinh T\,, \qquad x = X\, \cosh T\,.$$

What form does the line-element take in the (t, x) coordinates? Sketch the null geodesics in this new coordinate system.

Problem 5: Verify that the following constraint is consistent with the geodesic equation:

$$g_{\alpha\beta}(x)\, \frac{dx^\alpha}{d\tau}\, \frac{dx^\beta}{d\tau} = -c^2\,.$$

Problem 6: Killing vectors (optional). Derive Killing's equation, which was discussed in class:

$$\partial_\gamma g_{\alpha\beta}\, k^\gamma + g_{\alpha\gamma}\, \partial_\beta k^\gamma + g_{\beta\gamma}\, \partial_\alpha k^\gamma = 0\,,$$

where $\partial_\alpha k^\gamma \equiv \frac{\partial k^\gamma}{\partial x^\alpha}$. Show that this expression can be re-written as

$$\nabla_\beta k_\alpha + \nabla_\alpha k_\beta = 0 \quad \text{where} \;\; \nabla_\alpha k_\beta \equiv \partial_\alpha k_\beta - \Gamma^\gamma{}_{\alpha\beta}\, k_\gamma\,.$$

The following continuation of this question is doubly optional: Verify that although neither $\partial_\alpha k_\beta$ nor ${\Gamma^\gamma}_{\alpha\beta}\, k_\gamma$ is a tensor, the combination $\nabla_\alpha k_\beta$ is a tensor.

Using the fact that $k_\alpha v^\alpha$ is a scalar object, verify that when the vector index is raised the analogous covariant derivative is given by

$$\nabla_\alpha v^\beta \equiv \partial_\alpha v^\beta + {\Gamma^\beta}_{\alpha\gamma}\, v^\gamma \,.$$

Further verify that $\nabla_\alpha g_{\beta\gamma} = 0$.

Lecture 3

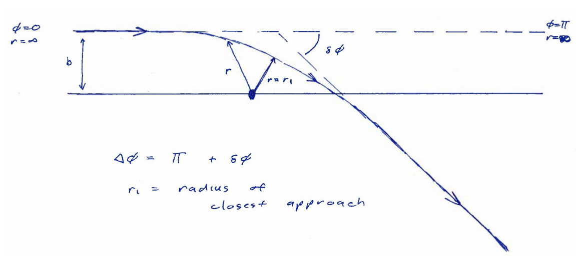
Problem 1: Gravitational Redshift. Imagine there are two observers, O_1 and O_2, at fixed positions in the Schwarzschild geometry, *i.e.,* at r_1 and r_2, as well as fixed angles. O_1 sends a photon to O_2 (along a radial geodesic). What is the change in the photon frequency measured by the two observers?

This effect was experimentally measured by Pound and Snider in a tower of height $h = 22.5$ metres on the surface of the earth. What was the relative change in the frequencies in their experiment? Here you can look up the mass and radius for the earth in Google or note that for this problem $M/r \ll 1$ and so you are in the regime covered by eq. (0.1) of Problem 1 for Lecture 2. (The original article describing this experiment was: Pound and Snider, *Physics Review Letters* **13** (1964) 539.)

Imagine $r_2 > r_1$, what happens in the limit that $r_1 \to 2M$?

Problem 2: Radial Orbits. Consider an observer falling along a radial geodesic, *i.e.,* $d\phi/d\tau = 0$. For simplicity, we will assume that he would be at rest at $r \to \infty$ – this picks a simple value for the conserved momentum associated with the "time" Killing vector $k^\alpha = \delta^\alpha_t$. Find the observers radius as a function of his proper time.

Problem 3: Gravitational Lensing. In the lecture, I discussed null geodesics in the Schwarzschild geometry, and, in particular, how the path of a photon coming in from infinity is bent by the central mass. Provide an expression to evaluate the deflection angle in the figure below.



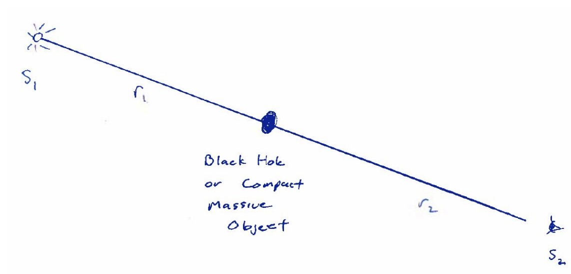
Recall that the "classical" answer in flat space would be that $\Delta\phi = \pi$. We are interested in the additional bending $\delta\phi$ induced by the curved spacetime geometry. The suggested approach is to use the expressions derived in class to derive an expression for $d\phi/dr$. The angle swept out by the orbit is then given by

$$\Delta\phi = 2\int_{r_1}^{\infty} dr\,\frac{d\phi}{dr}\,,$$

which is the radius of closest approach, *i.e.,* $dr/d\lambda = 0$ at $r = r_1$. You may find it helpful to re-express the integral in terms of the dimensionless variable $w = b/r$.

Use Mathematica or Maple to evaluate this expression numerically, *e.g.,* plot $\delta\phi$ as a function of $2M/b$. What is the result for $2M/b \ll 1$? Are there any other interesting limits?

Consider the illustration below showing a star S_1 and and observer S_2 diametrically on opposite sides of a black hole. Describe qualitatively what the observer will see? You may assume that the distances, r_1 and r_2, from the black hole to the star and the observer are large, *i.e.,* r_1, $r_2 \gg 2M$. Afterwards try looking up "Einstein ring" on Google.



Lecture 4

While these problems are based on today's lectures, I would rate them as optional. That is, if you want to focus on preparing for the exam on Saturday, I would suggest that you concentrate on the problems from Lecture 3 and earlier. You can come back to these problems later.

Problem 1: Einstein-Rosen bridge. Consider the Schwarzschild metric in geometrized units

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{dr^2}{1-\frac{2M}{r}} + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\,.$$

Show that after transforming the radial coordinate,

$$r = \rho\left(1 + \frac{M}{2\rho}\right)^2\,,$$

the line element becomes

$$ds^2 = -\left(\frac{1-\frac{M}{2\rho}}{1+\frac{M}{2\rho}}\right)^2 dt^2 + \left(1+\frac{M}{2\rho}\right)^4\left(d\rho^2 + \rho^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right)\,.$$

Find a further coordinate transformation which interchanges the region near $\rho = 0$ with the region near $\rho = \infty$, while leaving the metric coefficients in the same form. This is revealing the Einstein-Rosen bridge between two separate asymptotically flat regions hidden in the Schwarzschild geometry.

In this coordinate system, the last factor in the line element, *i.e.,* $d\rho^2 + \rho^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$, looks like flat three-dimensional Euclidean space written in polar coordinates. Rewrite the metric in terms of the usual Cartesian coordinates of flat space, *i.e.,* x, y, z. Now transform the coordinates once more with a boost. The resulting metric now describes a spherical black hole moving with a velocity v. What is the position of the horizon in the new boosted coordinate system? Describe the (intrinsic) geometry of a constant time slice of the new horizon. Is this cross-section still uniformly round?

Problem 2: First Law of Black Hole Mechanics. If we make a small increase in the mass in the Schwarzschild metric, show that the first law of black hole thermodynamics is satisfied, *i.e.,*

$$dM = T\, dS\,,$$

where temperature for a Schwarzschild black hole is given by

$$T = \frac{\hbar\, c^3}{8\pi k_B G\, M}$$

and in general the entropy is given by the Bekenstein-Hawking formula

$$S = \frac{k_B c^3}{4G\hbar}\, A_{horizon}\,.$$

Problem 3: Hawking Temperature. 'Wick rotate' the time coordinate in the Schwarzschild metric with $t = i\,\tau$ to produce the following Euclidean geometry:

$$ds^2 = \left(1 - \frac{2M}{r}\right) d\tau^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\,.$$

Show that $r = 2M$ is still only a coordinate singularity. In particular, consider the region very near $r = 2M$, *i.e.,* $r = 2M(1 + \varepsilon)$ with $\varepsilon \ll 1$, and with an appropriate coordinate change, show that the geometry of this region in the (r, τ) plane looks like flat space. However, the latter requires that τ is periodic with $\tau = \tau + \beta$. What is the value of β? It turns out that the Hawking temperature is given by $T_H = 1/\beta$. Relating the periodicity in Euclidean time with a temperature requires a longer explanation. If you are interested let me suggest that you start by looking up "thermal quantum field theory" on Wikipedia.