Integrability and other Simple Things

Pedro Vieira\textsuperscript{1,2} *

\textsuperscript{1}Perimeter Institute for Theoretical Physics, Waterloo, Ontario N2L 2Y5, Canada
\textsuperscript{2}ICTP South American Institute for Fundamental Research, IFT-UNESP, São Paulo, SP Brazil 01440-070

Abstract

This is a four lecture mini-course on Integrability for last year undergraduate students given at the Journeys into Theoretical Physics School 2016 at the ICTP-SAIFR in Sao Paulo.

Dear Participants of the School: Please let me know of any (small or big) typos and please feel free to suggest any sort of improvements. I am sure there are many since I did not really have time to go through the notes after writing them. Thanks!

*pedrogvieira@gmail.com
## Contents

1 Lecture 1. Wave Packets ........................................ 3
   1.1 Free Particle .................................................. 3
   1.2 Particle Scattering .......................................... 5
   1.3 Two Particle Scattering ................................... 7
   1.4 Three Particle Scattering ................................. 8
   1.5 Bouncing Ball (Bonus) ..................................... 10
   1.6 Exercises ....................................................... 12

2 Lecture 2. Integrability ......................................... 14
   2.1 Coleman-Mandula ............................................. 15
   2.2 Bosons with Contact Interactions ....................... 16
   2.3 Bound States .................................................. 17
   2.4 Quantum Spin-chains ...................................... 18
   2.5 Heisenberg Spin-Chain .................................... 18
   2.6 Magnons ......................................................... 19
   2.7 Exercises ....................................................... 20

3 Lecture 3. Bethe Ansatz for Quantum Spin Chains. ....... 22
   3.1 Magnon S-matrix ............................................... 22
   3.2 Three Magnon State – Integrability ..................... 22
   3.3 Bethe Ansatz Equations .................................... 23
   3.4 The Bethe Rapidities ........................................ 24
   3.5 Bethe Equations as Electrostatics ....................... 24
   3.6 Bethe Strings and Umbrellas – Math ................... 25
   3.7 Bethe Strings and Umbrellas – Physics ................ 28
   3.8 Exercises ....................................................... 28

4 Lecture 4. Anti-Ferromagnetic Physics. ..................... 30
   4.1 Another electrostatic picture. ............................ 31
   4.2 Anti-ferromagnetic Vacuum ................................ 32
   4.3 Holes and effective Bethe equations ................... 34
   4.4 AdS/CFT. String Theory and Integrability ............. 34
   4.5 Exercises ....................................................... 35
1 Lecture 1. Wave Packets

Consider a particle scattering past a small obstacle:

\[ e^{ikx} \rightarrow R(k)e^{-ikx} \rightarrow T(k)e^{ikx} \]

Figure 1: Scattering past a small obstacle.

To study this problem, we typically say that there is an incoming wave, a reflecting wave and a transmission wave. We then consider stationary solutions of Schrodinger equation $\hat{H}\psi = \frac{\hbar^2 k^2}{2m}\psi$ to find

\[ \psi(x) \simeq \begin{cases} e^{-ikx} + R(k)e^{ikx} & x < 0 \\ T(k)e^{-ikx} & x > 0 \end{cases} \]  

(1)

Where is the particle moving and scattering? This is just a stationary state. Its time evolution is trivial; we simply multiply it by a phase $\psi(x, t) = e^{i\frac{\hbar k^2}{2m}t}\psi(x)$ so that the probability density $|\psi(x, t)|^2 = |\psi(x)|^2$ and nothing special happens at time goes. So where is the particle moving with time and scattering?  

A way to clearly see what is going in is to consider wave packets. This is what this lecture is about. We will now work through a few simple examples.

1.1 Free Particle

Consider a single particle moving freely in empty space, without gravity or any other external force. Focus on a single space direction; the generalization to any number of dimensions is straightforward.

Classically the particle will follow a straight trajectory with constant velocity equal to the initial velocity of the particle.

In quantum mechanics, a plane wave with a single momenta is totally delocalized in space and is thus not describing a particle. A wave function corresponding to a space delta function is not describing a particle either since it would have maximally undefined momenta.

To describe a particle with a somehow well defined momenta and position we should use wave packets given by a superposition of plane waves with nearby momenta. Take for

---

1This holds for $x$ far from the obstacle. For $x$ at the obstacle the wave function is more complicated of course. It is its precise form, after all, which determines the reflection and transmission coefficients $R(k)$ and $T(k)$. Throughout this lecture we are after universal features of scattering and only care about what is happening far away from any scattering event.

2If the answer is obvious to you, you can probably skip this lecture or follow through for nostalgic reasons.
example a wave packet of the form

$$\psi(x,t) \propto \int dk \exp \left( -\frac{a^2}{2} (k-p)^2 + ikx - i \frac{\hbar}{2m} k^2 t \right)$$  \hspace{1cm} (2)

describing a linear superposition of plane waves with momentum around $k \simeq p$. Note that the $t = 0$ initial configurations described in the previous paragraph correspond to the extreme cases $a \to \infty$ and $a \to 0$. We can actually analytically evaluate the gaussian integral in (2) and get an analytic expression for the probability density $|\psi(x,t)|^2$,

$$|\psi(x,t)|^2 = \frac{1}{\sqrt{2\pi\Delta^2}} \exp \left( -\frac{1}{2\Delta^2} \left( x - \frac{\hbar p}{m} t \right)^2 \right), \quad \Delta = \frac{a}{\sqrt{2}} \sqrt{1 + \frac{t^2 \hbar^2}{a^4 m^2}}$$  \hspace{1cm} (3)

which we depict in figure 2.

At each moment in time, the maximum of the probability density is at $x - vt = 0$ with $v = \hbar p/m$. This is nothing but the classical trajectory with a velocity $v$. That trajectory is also represented by the solid red line in figure 2. This is how we make contact with the classical world.
The time dependent quantity $\Delta$ describes the spreading of the wave function. For example, we have

$$\Delta = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$$

for any time $t$. As time goes by this quantity increases and the gaussian (3) becomes broader (see exponent) and smaller (see pre-factor). If we start with a very small $a$ then the wave functions starts out very localized but the spreading is very fast. At large times the wave packet delocalizes and the classical approximation breaks down. We can estimate this transition as the time by which the position uncertainty $\Delta(t)$ is twice as big as its initial value $\Delta(0)$, that is when $\frac{\hbar^2}{a^2 m} \simeq 3$, see (3). For a macroscopic particle with $m = 1g$ and $a = 0.1cm$ we get $t \sim 10^{17}$ years which explains why we do not see a macroscopic particle spread out. For an electron taking $a = 10^{-8} cm$ as the typical size of an atom we get however $t \sim 10^{-16}$ seconds!

Finally let us note that of course we can reach all these physical conclusions (and more) by carefully analyzing the integrand (2). This is important since being able to compute the integral in (2) is a luxury which we can not hope to encounter as we move to more involved examples. So, for example, how could we see from the integrand in (2) that the particle will be localized along the red trajectory in figure 2?

Well, the integrand is a product of a Gaussian – which will localize the momenta at $k \simeq p$ – and a phase. Unless the that phase is stationary, it will oscillate a lot thus leading to a negligible wave function. We conclude that the wave function is located at the stationary value of the phase when the momenta is equated to $p,$ or

$$\frac{\partial}{\partial k} \left(i k x - i \frac{k^2 \hbar^2}{2m} t \right) \bigg|_{k=p} = 0$$

which indeed perfectly leads to the relation $x - vt = 0$ with $v = \hbar p/m$ which we had encountered before. Note again that here we reproduced this simple physical result without the need to explicitly evaluate any integral. This sort of argument will be recurrent in these lectures.

### 1.2 Particle Scattering

We can now revisit the scattering problem for which stationary waves are given by (1). It is pretty clear what to do. To get a particle which is localized both in time and in space we should not consider a state with a fixed (incoming) momentum but rather a wave packet. We make therefore a simple linear combination of states in (1) and time evolve them to get

$$\psi(x, t) \simeq \int e^{-\frac{a^2}{2} (k-p)^2} \times \begin{cases} 
T(k)e^{ikx - \frac{i k^2 \hbar^2 t}{2m}} & x > 0 \\
e^{ikx - \frac{i k^2 \hbar^2 t}{2m}} + R(k)e^{-ikx - \frac{i k^2 \hbar^2 t}{2m}} & x < 0
\end{cases}$$

(6)
Figure 3: When convoluted with a gaussian to create a realistic coherent state, the stationary state (1) turns into the wave packet (6). The latter describes a particle which is localized (in the sense of the previous section) precisely where we would expect it to be.

Now we use the argument at the end of the last section to see where this wave packet is localized. We want to look for the stationary values of the three phases showing up in (6),

\[
\begin{align*}
\text{incoming} & \quad e^{+i k x - \frac{1}{2} \frac{\hbar^2}{m} t} \quad \longrightarrow \quad + x - vt = 0 \quad \leftarrow \text{possible only if } t < 0 \text{ since } x < 0 \\
\text{reflected} & \quad e^{-i k x - \frac{1}{2} \frac{\hbar^2}{m} t} \quad \longrightarrow \quad -x - vt = 0 \quad \leftarrow \text{possible only if } t > 0 \text{ since } x < 0 \\
\text{transmitted} & \quad e^{+i k x - \frac{1}{2} \frac{\hbar^2}{m} t} \quad \longrightarrow \quad +x - vt = 0 \quad \leftarrow \text{possible only if } t > 0 \text{ since } x > 0
\end{align*}
\]

where we are taking here the velocity \( v = \hbar p/m \) to be positive. The particle will be localized at these stationary phases.

For negative times only the incoming wave stationary phase condition can be satisfied. So for \( t < 0 \) we have a particle approaching the obstacle from the left with a trajectory given by \( x - vt = 0 \) in perfect agreement with our classical physical intuition. For positive times, only the reflected and transmitted waves stationary conditions can be satisfied. So at positive times, after hitting the obstacle the particle can be on the left (i.e. reflected) with probability \( |R(k)|^2 \) and following the classical trajectory \( x \simeq -vt \) or it can be on the right (i.e. after passing by the obstacle) with probability \( |T(k)|^2 \) and following a classical trajectory \( x \simeq +vt \).\(^3\) In sum, we get precisely what we would expect for a particle hitting a small obstacle, as depicted in figure 3. We just clarified the small puzzle with which we started this lecture.

\(^3\)If we were to be more careful and take into account the reflection and transmission coefficients when computing the stationary phase conditions we would see that they lead to a slight time delay in this trajectories which is also to be expected.
1.3 Two Particle Scattering

Consider now two indistinguishable particles scattering against each other in one dimension. Before continuing let us remember that in one dimension elastic scattering is really quite simple. When two particle collide and conserve their energy and momentum so that the individual momenta and the two particles can at most be exchanged:

\[ P = k_1 + k_2 \]

\[ 2mE/\hbar^2 = k_1^2 + k_2^2 \]

Figure 4: In two dimensions we have \( 2mE/\hbar^2 = k_1^2 + k_2^2 \) and \( P = k_1 + k_2 = k'_1 + k'_2 \) where primed momenta are used for the momenta after the collision. As illustrated in this figure, this means that as a set \( \{k_1, k_2\} = \{k'_1, k'_2\} \). In other words, the final momenta are either equal to the initial ones or differ from those by a simple permutation.

Now let’s go quantum and describe the scattering process in terms of wave functions and wave packets. A stationary state is now a combination of an incoming wave and an outgoing wave,

\[ \psi(x_1, x_2) = e^{ik_1x_1+ik_2x_2} + S(k_1, k_2)e^{ik_2x_1+ik_1x_2} \] (8)

Note that both waves have the same energy and momenta since the individual momenta in them differ by a simple swap. Since the particles are indistinguishable it is enough to consider \( x_1 < x_2 \). If for some reason we insist in defining the wave function for \( x_1 > x_2 \) we simply make use of Bose statistics \( \psi(x, y) = \psi(y, x) \).

The coefficient multiplying the second plane wave – with the first wave normalized to one – is the called the S-matrix \( S(k_1, k_2) \). Now, normally we think of an S-matrix as describing the relative weight between incoming states prepares at \( t = -\infty \) and outgoing states received at \( t = +\infty \). But here we are dealing with stationary states so that nothing is really evolving non-trivially from minus to plus infinity. Of course, by now we know very well how to dispel such simple paradoxes using wave packets.

With the experience of the two previous examples, we now multiply the wave function above by two gaussians which will localize \( k_1 \approx p_1 \) and \( k_2 \approx p_2 \) and then look for the
time $t$

$$S(k_1, k_2) \times e^{i k_2 x_1 + i k_1 x_2}$$

$$t = 0$$

$1 \times e^{i k_1 x_1 + i k_2 x_2}$

Figure 5: The S-matrix $S(k_1, k_2)$ is the relative weight between the outgoing and incoming wave.

stationary phases of the two plane waves in the resulting wave packet. Let us take $v_1 = h p_1 / m > v_2 = h p_2 / m$. Then we have $x_1 - x_2 < 0$ and $v_1 - v_2 > 0$ which singles out a single plane wave depending on the sign of $t$:

incoming $e^{+i k_1 x_1 + i k_2 x_2 - \frac{k_1^2 \hbar^2}{2m} t - i \frac{k_1^2 \hbar}{2m} t} \rightarrow \begin{cases} x_1 - v_1 t = 0 \\ x_2 - v_2 t = 0 \end{cases} \quad \text{← possible only if } t < 0$

outgoing $e^{+i k_2 x_1 + i k_1 x_2 - \frac{k_2^2 \hbar^2}{2m} t - i \frac{k_2^2 \hbar}{2m} t} \rightarrow \begin{cases} x_2 - v_1 t = 0 \\ x_1 - v_2 t = 0 \end{cases} \quad \text{← possible only if } t > 0$

(9)

So at $t$ evolves from $-\infty$ to $+\infty$ the wave function goes from being dominated by the first plane wave function to being dominated by the second plane wave. As we explicitly see, the weight in this transition is precisely the $S$-matrix.

1.4 Three Particle Scattering

We considered one particle and then two particles. Three particles will teach us something new so we are going to consider this final case now. Let us prepare a the wave packet (at $t = 0$)

$$\psi(x_1, x_2, x_3) \simeq \int dk_1 \int dk_2 \int dk_3 \exp \left( -\sum_{j=1}^{3} \frac{\alpha^2}{2} (p_j - k_j)^2 + \sum_{j=1}^{3} i k_j (x_j - x_j^{(0)}) \right) \quad (10)$$

As before in this section we always have $x_1 < x_2 < x_3$. Note that we instead of simply $\sum_j k_j x_j$ we have $\sum_j k_j (x_j - x_j^{(0)})$ in the exponent. Mathematically, the $x_j^{(0)}$ are of course a...
Figure 6: (a) At \( t \approx 0 \) we prepare a wave packet as in (10). (b) With the momenta and with initial positions chosen in this figure, the two left-most particles will meet first. When they hit each other they are far from the rightmost particle so their scattering if effectively as described in the previous section. It will be governed by very same two-particle S-matrix \( S(k_1, k_2) \) which appeared before. (c) At some latter time the rightmost two particles scatter again swapping their momenta. (d) A last scattering event takes place after which the momenta are now re-organized. If in the beginning the momenta were ordered from left to right as \( k_j = \{ k_1, k_2, k_3 \} \) with \( k_1 > k_2 > k_3 \) now they scattered completely to the order \( k'_j = \{ k_3, k_2, k_1 \} \) so that \( k'_1 < k'_2 < k'_3 \). With this final ordering they will obviously no longer meet again.

trivial addition which can always be absorbed into the wave packet definition, that is into the gaussian dressing the plane waves. Physically, they are very important of course. At \( t \approx 0 \) this wave packet describes well separated particles as indicated in figure 6a.

Now we can easily evolve it in time as described in figure 6. In (schematic) formulae

\[
e^{i k_1 x_1 + i k_2 x_2 + i k_3 x_3} \overset{(a) \text{ to } (b)}{\longrightarrow} S(k_1, k_2) e^{i k_2 x_1 + i k_1 x_2 + i k_3 x_3} \overset{(b) \text{ to } (c)}{\longrightarrow} S(k_1, k_2) S(k_1, k_3) e^{i k_2 x_1 + i k_3 x_2 + i k_1 x_3} \overset{(c) \text{ to } (d)}{\longrightarrow} S(k_1, k_2) S(k_1, k_3) S(k_2, k_3) e^{i k_3 x_1 + i k_2 x_2 + i k_1 x_3}.
\]

(11)

Now, we prepared wave packets so that the particles meet in a pair-wise fashion. What if we carefully choose their initial momenta (or positions) such that at some latter time \( t_\ast \) the three wave packets meet each other at the same position \( x_\ast \). What happens after they meet? We only have energy and momentum conservation so a priori we have no reason to
Figure 7: When three particles meet momenta and energy conservation still leave plenty of room for what the final state could be as indicated in the figure. This is of course in stark contrast with figure 6 when the particles met in well separated regions (both in space and in time) so that the final momenta, as a set, were a simple re-ordering of the initial one. Here all we know if that their sum and the sum of their squares is the same as for the initial momenta but that still allows for infinitely many options, encoded here in the function $\phi(k_1', k_2', k_3')$ which a priori could be very complicated.

expect anything particularly simple after their collision. Does it mean that generically, after the three particles meet, the corresponding wave function will be an unpredictable mess as depicted in figure 7 and in particular will be strongly dependent on the details of the particular model under consideration?

Yep.

1.5 Bouncing Ball (Bonus)

As a last example let us consider a bouncing ball. A solid floor means $V(y) = +\infty$ for $y < 0$ while $V(y) = mg y$ for positive heights $y > 0$. Schrodinger equation with this potential is trivial to solve in terms of Airy functions. For the $n$-th excited state with energy $E_n$ we have

$$\psi_n(y) = \left(\frac{2gm^2}{\hbar^2}\right)^{\frac{1}{6}} \frac{1}{Ai(x_n)} \times Ai \left(\left(\frac{2}{g^2\hbar^2 m}\right)^{\frac{1}{3}} (mg y - E_n)\right)$$ (12)
where \( x_n = -E_n \left( \frac{2}{g^2 h^2 m} \right)^{\frac{1}{3}} \) is the \( n \)-th zero of the Airy function, see figure 8. Note that it is also the argument of the second Airy function in (12) as \( y \to 0 \) which ensures that the wave function vanishes at the floor as needed since the potential is infinite there. Note also that by definition of \( x_n \), the \( n \)-th excited wave function has \( n \) zeros as expected. The normalization prefactor before the \( \times \) in (12) ensures orthonormality

\[
\int_{-\infty}^{+\infty} dy \psi_n(y) \psi_m(y) = \delta_{nm}.
\] (13)

We can now hold a ball at some height \( h_{\text{initial}} \) and then simply drop it and see what happens. For that we start with a (sort of) Gaussian

\[
\psi_{\text{initial}}(y) = y e^{-\beta(y-h_{\text{initial}})^2}
\] (14)

where the \( y \) pre-factor is simply there so that \( \psi_{\text{initial}}(y = 0) = 0 \). Its time evolution is now trivial:

\[
\psi_{\text{initial}}(y) = \sum_{n=0}^{\infty} c_n \psi_n(y) \quad \longrightarrow \quad \psi(y, t) = \sum_{n=0}^{\infty} c_n \psi_n(y) e^{-iE_nt}
\] (15)

where the \( c_n = \int \psi_n(y) \psi_{\text{initial}}(y) dy \) since (13). In practice we can replace the sum over \( n \) by a sum up to a large integer \( \Lambda \) since the \( c_n \)'s will become very small for overlaps with wave functions whose energy is much larger than the classical energy of the ball, i.e. for \( E_n \gg mgh_{\text{initial}} \).

Figure 8: Airy function and its zeros.
We can now have some fun taking some initial wave function, decomposing it in wave functions, time evolving it, plotting its probability density and compare it with a classical bouncing ball. The result is illustrated in figure 9. The ball does bounce as expected for a classical ball (and it also spreads after many jumps as expected for a quantum entity). Physics is gorgeous.

Of course, it was key that we start with a wave packet as in (14) to get this nice picture. A single stationary state will not have any time evolution, by definition, so this is pretty much the same phenomena we already saw in all the previous sections.

1.6 Exercises

1. Check that (3) is properly normalized and derive it from (2). Establish (4).

2. Below (4) we plugged some numbers to find the spreading time for a macroscopic and a microscopic object. Not surprisingly, we obtained ridiculously large and small times respectively. Can you imagine some situation where the spreading time would be something in between, of a more human scale? (From a few micro-seconds to a few
3. Consider a particle coming from the left and encountering a large well of constant depth $V_0$ and length $2L$, that is $V(x) = -V_0 \theta(L^2 - x^2)$. How many plane waves are there in the stationary state now? What is their physical interpretation? At which times do they become relevant? In other words, repeat the discussion of section 1.2 for this case which now has three relevant regions.

4. What are the various scattering times in figure 6?

5. How would (11) change if the rightmost particles would scatter first? (This would be the case if, for instance, the initial position $x_2^{(0)}$ – and thus the full middle red trajectory in figure 10 – were shifted to the right closer to $x_3^{(0)}$.)

6. Verify (13). Generate yourself a plot like the one in figure (9). Hint: Plot the $c_n$ for $n = 0, \ldots, \Lambda$. You should see that most $c_n$’s are very small except for those corresponding to an energy corresponding to the potential energy related to the wave packet bump you start with, see for example figure 10. If this is not the case it means your $\Lambda$ is not big enough. If this is the case your plots should look nice.
Consider a (one dimensional) parallel universe where on top of conservation of momentum $Q_1 \equiv \sum k_j$ and energy $Q_2 \equiv \sum k_j^2$ we have more symmetry and in particular a new hidden charge

$$Q_3 = \sum k_j^3. \quad (16)$$

(As we will see below, there are indeed interesting physical models, realized in nature, where such hidden charges do exist so this is not such an esoteric idea.) How would life in such universe be?

First of all, the mess in figure 7 would be gone! Indeed, if we have an extra conserved charge such as (16) the final momenta of the three particles would have to be the same (up to a simple re-shuffling) as the initial momenta.

At this moment we could think: Well, the mess in the three particle case is gone but we just postponed the problem. For four particles we will get a mess again... Here our inner physical intuition should sound an alarm: If the mess if gone for three particles something deep is going on. Probably there is a nice physical reason for this which will also ensure that for four and more particles all is good. To make this intuition more rigorous lets pretend to be an experimentalist who carefully prepares three wave packets in the past and collects them again in the future. The experimentalist has no access to what happens at intermediate times; it is effectively a black box. Such as experimentalist would be very surprised when measuring the very same set of momenta in the future which he prepared in the past (rather than just a mess). His conclusion would be: *Hum, it seems like inside this box the particles effectively behaved as if they had scattered in a sequence of pairwise events.* Indeed, this would definitely be the simplest explanation. If this is correct then why wouldn’t it be the same for four particles?

Armed with the wave packet technology we played with yesterday we can put sharp formulae on top of these handwaving arguments. Consider the wave packet (2) and lets act on with the various symmetry transformations. In momentum space the action of $Q_n$ is trivial. We simply multiply the integrand in (2) by $\exp(i\beta k^n)$. Where is the particle after this? Well, we can simply go to (5) and modify it due to this new contribution to the phase to get

$$0 = x - pt + n\beta p^{n-1} \quad (17)$$

Note that for $n = 1$ this can be seen as a simple shift in $x$ (as expected since $Q_1$ is just the momentum) while for $n = 2$ this induces a shift in $t$ (which is again expected since $Q_2$ is the energy). On the other hand, for $n = 3$ the wave packet is shifted by a momentum-dependent amount. Let us now act with these symmetry on the process in figure 7. Since we shift each wave packet by a different amount (since they all have different momenta) they will no longer meet! So in a world with such symmetries we conclude that the probability amplitudes associated to the processes in figure 11a and 11b are the same. In such a world the $3 \rightarrow 3$ scattering process factorizes into a sequence of two body-scattering events. In fact, we can

---

4We are setting $\hbar = m = 1$ for simplicity here (in particular $v = p$ with these units).
simply take (11) to read of the three-body S-matrix

\[ S(k_1, k_2, k_3) = S(k_1, k_2)S(k_1, k_3)S(k_2, k_3). \]  

(18)

Similarly, the n-body S-matrix would be be given by a totally factorized process as well with

\[ S(k_1, \ldots, k_n) = \prod_{i<j} S(k_i, k_j). \]  

(19)

In such universe, scattering is simple. It suffices to know the $2 \rightarrow 2$ scattering amplitude and then use it as a building block to build any other S-matrix element involving more particles. Such theories are known as integrable theories.

### 2.1 Coleman-Mandula

The previous discussion was about 1+1 dimensions. What about higher dimensions? We can also generate wave packets there easily. Suppose we conceive, in that case, some higher charge like $Q_3$. It will again shift each wave packet by a momentum dependent amount but now there is a key difference. Because the wave packets can leave the paper, after acting with such symmetries they will typically no longer interact! By acting with a big transformation parameter we can thus separate the wave packets putting them in different galaxies, see figure 12. In other words, we get no scattering at all. So it seems like if one tries to be greedy and add new higher symmetries to our quantum theory we end up with a free theory.
Huge impact parameter = 1

Figure 12: Higher charges in higher dimensions relate the probability amplitude associated to a process where particles meet to one where they do not and where the corresponding $S$-matrix is 1. Hence, such theories need to be free.

if we are in higher dimensions (and an integrable theory in two dimensions). This is the essence of the Coleman-Mandula theorem.

2.2 Bosons with Contact Interactions

So far all this was very abstract. What is the simplest example of an integrable theory? Probably delta-function bosons would be the simplest. Its Hamiltonian reads

$$\hat{H} = -\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} + c \sum_{j \neq k} \delta(x_j - x_k)$$  \hspace{1cm} (20)

and its stationary state wave functions $\psi(x_1, \ldots, x_n)$ are simple sums of $n!$ plane waves corresponding to all possible permutation of the momenta. For example, for three particles they read

$$\psi(x_1, x_2, x_3) = e^{ik_1 x_1 + ik_2 x_2 + ik_3 x_3} + S(k_1, k_2) e^{ik_2 x_1 + ik_1 x_2 + ik_3 x_3}$$
$$+ S(k_1, k_2) S(k_1, k_3) e^{ik_2 x_1 + ik_3 x_2 + ik_1 x_3} + S(k_1, k_2) S(k_2, k_3) e^{ik_3 x_1 + ik_2 x_2 + ik_1 x_3}$$
$$+ S(k_2, k_3) e^{ik_1 x_1 + ik_3 x_2 + ik_2 x_3} + S(k_1, k_3) S(k_2, k_3) e^{ik_3 x_1 + ik_1 x_2 + ik_2 x_3}$$ \hspace{1cm} (21)

The generalization to more particles is obvious. It often helps to draw a simple cartoon for each term in the wave function. For example for the wave function we just wrote we would have
The two particle S-matrix is computed by carefully solving Schrodinger equation matching the wave function across the various delta functions. The result turns out to be quite simple:

\[ S(k_1, k_2) = \frac{k_1 - k_2 - ic}{k_1 - k_2 + ic}. \] (22)

### 2.3 Bound States

In the previous section we found a model of bosons which move freely except when they are right on top of each other when they interact via a contact interaction (a \( \delta \)-function). It turns out that there is quite a lot of physics within such a simple model. Here, as an example, we shall see that these particles can sometimes form molecules by binding together.

Take a two particle wave function

\[ \psi(x_1, x_2) = (k_1 - k_2 + ic) \times e^{ik_1 x_1 + ik_2 x_2} + (k_1 - k_2 - ic) \times e^{ik_2 x_1 + ik_1 x_2} \] (23)

We have multiplied the full wave function by the S-matrix denominator to simplify the discussion below. Can \( k_1 \) and \( k_2 \) take complex values? Yes, but only if one of the pre-factors multiplying the exponentials in (23) vanishes. The reason is that when the momenta are complex, one of the two plane waves will not converge at infinity which means that we can only accept that solution if its pre-factor vanishes.

We can assume \( \text{Im}(k_2) > \text{Im}(k_1) \) without loss of generality since the particles are identical. Then the first exponential is fine since we always have \( x_2 > x_1 \) (we are dealing with indistinguishable particles so this is enough). The second exponential, on the other hand, blows up at infinity so that we must kill its pre-factor setting \( k_1 - k_2 = ic \). Now if \( c > 0 \) we reach a contradiction since we assumed \( k_2 \) had a bigger imaginary part. So we conclude that

- For positive \( c \) we should take \( k_1 \) and \( k_2 \) to be real.
- For negative \( c \) we can take \( k_1 = p + ic/2 \) and \( k_1 = p - ic/2 \). This kills the second exponential in (23) so that the wave function becomes

\[ \psi(x_1, x_2) \sim e^{ip(x_1 + x_2) - (c)(x_2 - x_1)} \] (24)

We see that the wave function moves with center of mass momentum \( p \) but decays as the second particles moves away from \( x_1 \). For obvious reasons this is what is called a bound-state.

Note that \( c > 0 \) means a positive delta-function in (20) hence a repulsive potential while a negative \( c \) means an attractive force. Hence the conclusions we just obtained seem very sensible. In particular, the larger (and negative) \( c \) is stronger is the binding of the two particles are.
2.4 Quantum Spin-chains

We will conclude this lecture by introducing a second integrable model. We will start exploring its physics tomorrow. The model is that of a so-called quantum spin chain where we have a spin $|\uparrow\rangle$ or $|\downarrow\rangle$ at each site of a long chain. A random state could be

$$|\psi\rangle = \langle \downarrow\downarrow\uparrow\downarrow\downarrow\uparrow\downarrow\downarrow\downarrow\ldots\rangle$$

$$= (\sigma_3^+ \sigma_8^+ \sigma_9^+ \ldots) \left( |\text{ferro-magnetic vacuum}\rangle \equiv |\Omega\rangle = \langle \downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\ldots\rangle \right)$$

$$\equiv |3,8,9,\ldots\rangle \equiv |\text{ferromagnetic state with spin flip excitations at positions } 3,8,9,\ldots\rangle$$

where the Pauli matrices $2\sigma_1^+ = \sigma_1^1 + i\sigma_2^2$ acts as a raising operator flipping a spin at position $n$. It acts as $\sigma^+ |\downarrow\rangle = |\uparrow\rangle$ and $\sigma^+ |\uparrow\rangle = 0$. We identify the first and last spins. In other words, we are dealing with a spin chain which is a ring.

Such chains could arise from a model with a lattice where electrons move. If we put in this lattice as many electrons as lattice sites and if the electrons repel each other then each site will be occupied by a single electron which can have spin up or down. Then the relevant states would be states of the form (25). Of course, if we heat up the system a lot the electrons would start hopping around and we would need a larger Hilbert space with four states per lattice: empty, spin up electron, spin down electron and two electrons (one spin up and another spin down which is of course not forbidden by Pauli’s exclusion principle). Such chains can also describe some interesting physical compounds such as KCuF$_3$ crystal whose magnetic dipole interactions are effectively one-dimensional.

The Hilbert space for a spin chain of length $L$ is huge, of dimension $2^L$. So an Hamiltonian for a spin chain will be a big $2^L \times 2^L$ matrix. In a computer we will typically be able to do about twelve sites. The problem is of exponential complexity so it is typically very hard to increase the size of the system, even by a single spin!

2.5 Heisenberg Spin-Chain

Integrability will come to the rescue. It turns out that the simplest nearest neighbour spin chain Hamiltonian we could write is exactly solvable precisely because it is secretly endowed with remarkable higher charges akin to the $Q_3$ introduced above which ensure the model is integrable. The Hamiltonian is the so-called Heisenberg spin chain,

$$\hat{H} = \frac{J}{2} \sum_{n=1}^{L} (\mathbb{I}_{n,n+1} - \vec{\sigma}_n \cdot \vec{\sigma}_{n+1}) .$$

(26)

If $J$ is positive the vacuum spins want to be aligned so that the vacuum is the ferromagnetic vacuum presented above while if $J$ is negative the vacuum is the (much more complicated and probably more interesting for condensed matter theorists) anti-ferromagnetic vacuum. In the last lecture we will consider $J < 0$; until then we focus on the ferromagnetic case where $J > 0$. 

18
The Hamiltonian density can be thought of as a 4 × 4 matrix acting on the Hilbert space \{↑↑, ↑↓, ↓↑, ↓↓\} of sites \(n\) and \(n + 1\). In this basis \(I_{n,n+1} - \vec{\sigma}_n \cdot \vec{\sigma}_{n+1}\) is given by
\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
- \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}
- \begin{pmatrix}
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0
\end{pmatrix}
- \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} = 2 \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} - 2 \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

Where in the second line we recognize (twice) the identity operator minus (twice) the permutation operator which permutes spins,
\[
\mathbb{P} |↑↑⟩ = |↑↑⟩, \quad \mathbb{P} |↑↓⟩ = |↓↑⟩, \quad \mathbb{P} |↓↑⟩ = |↑↓⟩, \quad \mathbb{P} |↓↓⟩ = |↓↓⟩.
\]

In sum, we have an equivalent representation of the Heisenberg Hamiltonian as
\[
\hat{H} = J \sum_{n=1}^{L} (I_{n,n+1} - \mathbb{P}_{n,n+1}) .
\]

As we will see, this simple looking Hamiltonian has in it a tremendous amount of fascinating physics.

### 2.6 Magnons

When \(J > 0\) the vacuum of (29) is simply the ferromagnetic vacuum
\[
|Ω⟩ = |↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓ ...⟩ .
\]

Note that it is particularly simple to see that this state is an eigenstate of (29) with zero energy. Indeed, at each pair of sites the Hamiltonian density yields zero acting on this state since both the identity and the permutation operator give the same result when acting on identical spins.

What about its excitations? Well, they will be obtained by flipping some of the spins. Each spin flip is called a magnon excitation. These magnons move in the chain since the permutation operator sends this excitation to one of its neighbours! Indeed, since the Hamiltonian is translation invariant, to find a solution with a single spin flip we should simply give it some momenta as
\[
|1 \text{ magnon}⟩ = \sum_n e^{ipn} |\underbrace{↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓ ...}_{n-1}⟩ = \sum_n e^{ipn} \sigma_n^+ |Ω⟩ = \sum_n e^{ipn} |n⟩
\]

Acting with the Hamiltonian on \(|n⟩\) almost gives zero since all pairs except the sites \(n - 1, n\) and \(n, n + 1\) are trivially killed by \(\hat{H}\) for the same reason and the vacuum. So we only need to consider \(J(I_{n-1,n} - \mathbb{P}_{n-1,n}) + J(I_{n,n+1} - \mathbb{P}_{n,n+1})\) to conclude that
\[
\hat{H}|n⟩ = J|n⟩ - J|n - 1⟩ + J|n⟩ - J|n + 1⟩
\]
so that

\[ \hat{H}|1 \text{ magnon}\rangle = J \sum_n e^{ipn} (2|n\rangle - |n - 1\rangle - |n + 1\rangle) \tag{33} \]

\[ = J \sum_n (2e^{ipn} - e^{ip(n+1)} - e^{ip(n-1)} |n\rangle) \]

\[ = J (2e^{ip} - e^{-ip}) \sum_n e^{ipn} |n\rangle \]

\[ = 4J \sin^2 \left(\frac{p}{2}\right) |1 \text{ magnon}\rangle, \]

We see that a magnon behaves very much like a particle. It is an excitation which hops along the spin chain with momentum \( p \) and energy

\[ \varepsilon(p) = 4J \sin^2 \left(\frac{p}{2}\right). \tag{34} \]

Note that for very small momentum we have very long wave-length so that the details of the lattice should be washed out. Indeed, in this limit we get \( \varepsilon(p) \sim p^2 \) as expected for a non-relativistic particle in the continuum. This is not surprising since

\[ \hat{H} \sum_n \psi(n)|n\rangle \propto \sum_n (2\psi(n) - \psi(n - 1) - \psi(n + 1))|n\rangle \tag{35} \]

which is nothing but the discrete version of \( \partial^2 \psi/\partial x^2 = \lim_{\epsilon \to 0} \frac{1}{\epsilon^2} (2\psi(x) - \psi(x - \epsilon) - \psi(x + \epsilon)) \).

The next step is clear: To understand all excited states we start with the ferromagnetic vacuum and add not one but several magnons. They will interact with each other and lead to very interesting physics and mathematics. This is what we shall turn to next.

### 2.7 Exercises

1. As in the first lecture, convolute the exact stationary state (21) with appropriate wave packets and study its time evolution. Explain how to make contact with what add in figure 6 and also in problem 5 of the previous section.

2. Compare the energy of the bound-state encountered in section 2.3 to the energy of two particles with real momentum with the same total momentum as the bound-state. Which one is larger? Does it make sense?

3. We can also form more complicated molecules in the \( \delta \)-function boson model, with more than two particles binding together. Work this case out.

4. Suppose we put the three particle wave function (21) in some large circle of length \( L \). Then periodicity of the wave function, compatible with the ordering we choose for identical particles would be the condition \( \psi(x_1, x_2, x_3) = \psi(x_2, x_3, x_1 + L) \). What equations on the momenta does this yield? Can you attach some simple physical meaning to those equations?
5. Show that the Heisenberg Hamiltonian (26) has $SU(2)$ symmetry, that is show that it commutes with all components of the total spin operator $\vec{S} = \frac{1}{2} \sum_{n=1}^{L} \vec{\sigma}_n$.

6. It is of course possible to find (29) in a more sophisticated without any explicit matrix algebra. Show that $I_{n,n+1} - \vec{\sigma}_n \cdot \vec{\sigma}_{n+1} = 4 I_{n,n+1} - 2 \left( \frac{\vec{\sigma}_n + \vec{\sigma}_{n+1}}{2} \right)^2$. What is the action of the second term on a symmetric state (i.e. in a state of spin 1)? What is the action of the second term on a state on an antisymmetric state (i.e. in a state of spin 0)? Use these results to write $I_{n,n+1} - \vec{\sigma}_n \cdot \vec{\sigma}_{n+1}$ in terms of the projectors into symmetric and anti-symmetric states $P = \frac{1}{2} (I \pm \mathbb{P})$. (Before doing so understand why these are indeed projectors (into those spaces).) In this way you should find (29) without ever using any matrix manipulations.

7. Is $\sum_{n<m} e^{ipn+ikm} |n,m\rangle$ a good 2-magnon eigenstate of the Hamiltonian? Why/why not?

Here we continue our analysis of the Heisenberg chain

$$\hat{H} = J \sum_{n=1}^{L} (\mathbb{I}_{n,n+1} - \mathbb{P}_{n,n+1}) . \quad (36)$$

In the last lecture we saw that a single magnon \(|1 \text{ magnon}\rangle = \sum_{n} e^{ipn} |n\rangle\) behaves like a particle hopping around the spin chain with momentum \(p\) and with energy \(\varepsilon(p) = 4J \sin^{2}(p/2)\). Let us explore multiple spin flips now.

### 3.1 Magnon S-matrix

Next let us consider two excitation. This is quite interesting since the two magnons will now also scatter between themselves. Acting on the state

$$|\psi\rangle = \sum_{n<m} \psi(n,m)|n,m\rangle , \quad (37)$$

with the Heisenberg Hamiltonian and imposing \((\hat{H} - E)|\psi\rangle = 0\) yields

$$E/J \psi(n,m) = 4\psi(n,m) - \psi(n+1,m) - \psi(n-1,m) - \psi(n,m+1) - \psi(n,m-1) \quad (38)$$

for \(m > n + 1\) and

$$E \psi(n,n+1) = 2\psi(n,n+1) - \psi(n-1,n+1) - \psi(n,n+2) \quad (39)$$

when the spin flips meet. Any superposition of plane waves

$$\psi(n,m) = e^{ikn+ipm} + S(k,p)e^{ipm+ikm} \quad (40)$$

solves the first equation describing the free propagation and gives \(E = \varepsilon(k) + \varepsilon(p)\) while the second condition governs the scattering between the magnons and fixes

$$S(k,p) = \frac{1}{2} \frac{\cot \frac{k}{2} - \frac{1}{2} \cot \frac{p}{2} - i}{\frac{1}{2} \cot \frac{k}{2} - \frac{1}{2} \cot \frac{p}{2} + i} \quad (41)$$

This is the magnon S-matrix!

### 3.2 Three Magnon State – Integrability

Let us then consider three particle states where integrability can play a key role. Indeed it is simple to check (but perhaps not fun) that

$$|\psi\rangle = \sum_{n_1<n_2<n_3} \psi(n_1,n_2,n_3)|n_1,n_2,n_3\rangle ,$$
Figure 13: As a particle is carried around the ring it scatters with all others. Periodicity of the wave function is the condition that the total acquired phase must be trivial. This should be true for each of the particles and gives rise to the $N$ Bethe equations which quantize the momenta of the $N$ magnons.

with

$$\psi(n_1, n_2, n_3) = \phi_{123} + \phi_{213} S_{12} + \phi_{132} S_{23} + \phi_{312} S_{31} S_{12} + \phi_{321} S_{23} S_{13} S_{12},$$

$S_{ij} \equiv S(p_i, p_j)$ and $\phi_{ijk} \equiv \exp(ip_i n_1 + ip_j n_2 + ip_k n_3)$ is an eigenstate of $\hat{H}$ with energy $E = \varepsilon(p_1) + \varepsilon(p_2) + \varepsilon(p_3)$. The wave function for many magnon states are obtained following the obvious pattern in (42) as described in the previous lecture for delta function bosons. This ansatz for the wave function is of course far from obvious and was the key observation of Bethe in 1931 to realize it would work. As we saw in the previous section such ansatz will generically work if there are a lot of extra conserved charges which is of course not obvious when we look at a given Hamiltonian. The ansatz (42) and its multi-magnon generalization goes by the name as the Bethe ansatz.

3.3 Bethe Ansatz Equations

As we saw, because we can always measure the energy of the state when the magnons are separated, that the energy of an $N$ magnon state is simply given by the sum of the energies of each constituent magnon

$$E = \varepsilon(p_1) + \cdots + \varepsilon(p_N).$$

(43)

Of course, this cannot be the full answer to our spin ring problem. If the ring is finite then we are dealing with a finite quantum system which must have a discrete spectrum so we cannot simply take any $p_j$ we want and simply plug them in (43). Instead we must impose that the multi-magnon wave functions are periodic which in our notation – where $n_1 < \cdots < n_N$ avoid over counting of the identical magnon excitations – simply reads

$$\psi(n_1, n_2, \ldots, n_N) = \psi(n_2, \ldots, n_N, n_1 + L)$$

(44)

The periodicity of the wave function imposes a set of $N$ quantization conditions for the $N$ momenta,

$$e^{ip_j L} \prod_{k \neq j}^N S(p_k, p_j) = 1, \quad j = 1, \ldots, N$$

(45)
which are called Bethe equations.

The physical meaning of this equation is the following: if we carry a magnon with momentum \( p_j \) around the circle, the free propagation phase \( p_j L \) plus the phase change due to the scattering with each of the other \( N - 1 \) magnons must give a trivial phase, see figure 13. We shall denote by Bethe eigenstates those Bethe states whose momenta are quantized as in (45).

Let us stress again: the fact that the simple wave functions described above diagonalize the Heisenberg Hamiltonian is absolutely remarkable and non-trivial. A generic spin chain Hamiltonian will not lead to an integrable theory, the scattering will not factorize into two-body scattering events and the set of momenta of multiparticle states will not be conserved. Hence, in general, the problem will be of exponential complexity and the best we can do is diagonalize small spin chains with a computer. The Bethe ansatz reduces this problem to a polynomial one. For example, the spectrum problem is completely solved by the simple set of algebraic equations (45).

### 3.4 The Bethe Rapidities

The Bethe rapidities \( u_j \) are a convenient parametrization of the momenta \( p_j \) given by

\[
 u = \frac{1}{2} \cot \frac{p}{2}, \quad e^{ip} = \frac{u + i/2}{u - i/2}. \tag{46}
\]

With this parametrization the Bethe equations (45) simplify dramatically into

\[
 1 = \left( \frac{u_j + i/2}{u_j - i/2} \right)^L \prod_{k \neq j} u_j - u_k - i, \quad E/J = \sum_{j=1}^{N} \frac{1}{u_j^2 + 1/4}. \tag{47}
\]

We should pause to appreciate how much better off we are. To find the full spectrum of a quantum spin chain Hamiltonian we simply need to solve these simple looking algebraic equations!\(^5\)

### 3.5 Bethe Equations as Electrostatics

To study the solutions to (47) it is convenient to take the log of these equations. Obviously

\[
 e^{ix} = e^{iy} \Rightarrow x = y + 2\pi n \tag{48}
\]

so that for each \( j = 1, \ldots, M \) we have a possible choice of the branch log parameterized by an integer \( n_j \). We can thus write the Bethe equations as

\[
 F(u_j) - 2\pi n_j + \sum_{k \neq j} f(u_j - u_k) = 0 \tag{49}
\]

\(^5\)We can of course bring them to the same denominator so they are more than algebraic equations; they are polynomial equations (of many variables of course).
where
\[ F(u) = \frac{L}{i} \log \frac{u - i/2}{u + i/2}, \quad f(u) = \frac{1}{i} \log \frac{u + i}{u - i}. \]

(50)

In the definitions of \( f \) and \( F \) we chose the branch of the log in such a way that these functions decay for large real arguments, see figure 14.

We can now use our physical intuition to understand where the Bethe roots will organize themselves in the complex plane. To do so we think of (49) as the static equilibrium condition for the positions \( u_j \) of \( M \) particles. In this language \( F(u_j) \) is an external force felt by each particle located at \( u_j \) and \( -2\pi n_j \) is a constant force exerted on particle \( j \). Finally \( f(u_i - u_k) \) is an interaction force by particle \( k \) on particle \( j \).

3.6 Bethe Strings and Umbrellas – Math

Let us develop this intuition. Suppose we choose all mode numbers to be the same. Then, without a repulsive force all particles would go to the same potential equilibrium position which we can easily find by solving \( F(u) - 2\pi n = 0 \). But then there is the interaction force \( f \) whose vector field is represented in figure 15. We see that this force attracts in the \( x \) direction (as we knew already from figure (14) and repels in the complex \( y \) direction. So the particles will orient themselves vertically. This shape will then bend and form an umbrella like shape as depicted in figure 16a. This is again clear from the static equilibrium picture: the roots close to the real axis are closer to the origin and therefore feel a larger repulsive external force. Thus they will be pushed to the right more that the endpoints of the cut.

For the generic situation where the Bethe roots are grouped into \( K \) sets of roots sharing the same mode numbers, we obtain \( K \) umbrella cuts in the complex plane as represented in
Figure 15: Vector plot of \((Re(f), Im(f))\) in the complex \(u\) plane. This interaction force squeezes horizontally and stretches vertically thus leading to complex solutions.

Figure 16b for \(K = 2\). (An interesting extreme case is when all particles have different mode numbers. Then \(K = N\) and we have a solution where all roots are real.)

A natural question one might pose is how bent are these umbrellas? It depends dramatically on where we are trying to put these cuts. Recall that the position \(u\) (interception with the real axis) of the cut is given in a first approximation by

\[ \frac{L}{i} \log \frac{u + i/2}{u - i/2} = 2\pi n, \]  

(51)

i.e. it is dictated by the choice of the mode number \(n\). If \(L\) is large there are two natural choices for \(n\):

\[ n \sim L, \text{ Choice A}, \]

\[ n \sim 1, \text{ Choice B}. \]

Choice A is the most common in the condensed matter literature. When \(n \sim L\) the equilibrium position is given by \(u^* \sim 1\) and the umbrellas are almost absolutely straight vertical lines with Bethe roots separated by \(i\). To see this consider a configuration with two Bethe
Figure 16: In the $SU(2)$ electrostatic pictures the particle attract in the horizontal direction and repel in the vertical direction. They also feel an external force with opposite behavior. Therefore the cuts will orient themselves vertically. When there is a single cut we find the umbrella shaped cuts as depicted in the left. This is because the roots close to the real axis feel the horizontal repulsive external force more strongly than the roots in the tails. For two cuts we obtain the picture in the right. The roots attract one another horizontally and therefore the middle of both cuts want to approach each other leading to the observed deformation of the cut to the right.

Let us write $u_{1,2} = u \pm vi$ where $v > 0$. Then it is clear that

$$\left| \left( \frac{u_1 + i/2}{u_1 - i/2} \right)^L \right| \gg 1 \quad (53)$$

where by $\gg 1$ we mean exponentially divergent (suppressed) in the parameter $L$. This means that the roots $u_1$ and $u_2$ must be such that the r.h.s of the first equation in (52) is also exponentially large. Thus we must have

$$|u_1 - u_2 - i| \ll 1. \quad (54)$$

Equation (54) means that the two Bethe roots in the $SU(2)$ chain are very rigidly bound and their separation is precisely $u_1 - u_2 = i$ with exponential precision. The same analysis could
be carried for more than two roots and the conclusion would be that we can have bound states of $M$ particles separated by $i$ up to exponentially suppressed corrections:

$$ u_j = u + ij , \ j = -\frac{M}{2}, \ldots, \frac{M}{2}. $$

(55)

Thus we see that the $SU(2)$ umbrellas in case $A$ are not bent at all.

Next we consider case $B$. The mere existence of another scenario might seem odd at first since the argument leading to the conclusion that the roots must be separated by $i$ seems spotless. It is not so. The key difference between case $A$ and $B$ is that in the former the equilibrium position following from (51) is $u \sim 1$ whereas in the latter we have $u \sim L$. In case $B$ equation (53) does not follow and we have instead

$$ \left( \frac{u_1 + i/2}{u_1 - i/2} \right)^L \simeq \exp \left( \frac{L}{u} \right) = \mathcal{O}(1). $$

(56)

Therefore the r.h.s of the first equation in (52) no longer needs to explode and we can have complex roots which are not rigidly separated by $i$.

### 3.7 Bethe Strings and Umbrellas – Physics

The umbrellas in scenario $B$ correspond to long wavelength excitations around the ferromagnetic vacuum. Indeed, for such large $u$’s we have $E \sim 1/L^2$ or, if we have $O(L)$ of them, $E \sim 1/L$ which is still very small. They govern the low energy dynamics around the ferromagnetic vacuum. Basically, in this limit we can imagine a coherent state describing how the spins change slowly along the chain by means of a slow varying field $\vec{n}(x)$. This field can have several Fourier modes excited with various Fourier amplitudes.\(^6\) The number of umbrella and their sizes correspond to these two quantum numbers.

The Bethe strings of scenario $A$, on the other hand, correspond to bound-states in the sense we encountered yesterday for the delta function bosons. These have $E = \mathcal{O}(1)$. The existence of bound-states in the Heisenberg chain is physically expected. Imagine a chunk of spins up moving close together on top of the ferromagnetic vacuum with all spins down as depicted in figure 17. This should be energetically very favourable since inside this block we also have a ferromagnetic vacuum and thus no energy cost. All energy cost is localized at the border. Based on this picture we expect magnons to attract and form bound-states.

### 3.8 Exercises

1. Derive (38), (39) and (41).

\(^6\)This is an oversimplification of sorts. The effective model describing the long wave-length physics in in fact quite non-linear so we do not have simply a set of independent Fourier modes. (The counterpart in the Bethe ansatz language is that the various umbrellas do interact with each other as clearly seen in figure 16)b. Still, there is an analogue of Fourier mode and Fourier amplitude which justify the analogy in the text fine.
2. Carefully check that (44) does lead to (45) for the cases $N = 2$ and $N = 3$. This amounts to carefully seeing how the various exponentials in the Bethe ansatz match when we compare the left and right hand sides in (44).

3. Verify (47).

4. We encountered Bethe strings in this lecture and bound-states of delta-function bosons in the previous lecture. Show that they can both be understood in the same way. Namely, show that Bethe strings which we found by analyzing Bethe equations can also be explained purely by looking at the wave function and asking for some non-normalizable exponentials to vanish – like we did for the delta-function bosons. Conversely, construct the Bethe equations for the delta-bosons and show that the bound-states found there can be understood as Bethe strings – in the same sense as we found in (55) for the magnons.

5. Repeat the analysis of the solutions of Bethe equations of section 3.6 for the so-called $SL(2)$ Bethe equations described by

$$1 = \left( \frac{u_j + i/2}{u_j - i/2} \right)^L \prod_{k \neq j}^{N} \frac{u_j - u_k + i}{u_j - u_k - i}, \quad (57)$$

which differ by small but very important sign from the $SU(2)$ equations (47). Hint: You should convince yourself that there are actually no complex solutions in this case! What would this tell you about the nature of the interaction between particles in whatever model these equations are describing?

In the last two lectures we understood the nature of the magnon as summarized in figure 18. The magnon is the fundamental excitation around the ferromagnetic vacuum. It flips a spin $-1/2$ into a spin $+1/2$ so that it carries spin $1 = 1/2 - (-1/2)$. We computed its dispersion relation in lecture 2 and its S-matrix in lecture 3.

In this lecture we shall focus on anti-ferromagnetic vacuum. This will be the vacuum of the Heisenberg chain for $J < 0$. It is a very complicated singlet state. It is not just a Neel state. What are its excitations? What is their spin, their dispersion, their S-matrix etc? This is what we address here. A spoiler is summarized in figure 18.

Note something very surprising in the spinon card: it is a spin 1/2 excitation. Any change in the Hilbert space involves changing a spin down to a spin up or vice-versa – thus always changing the total spin by 1. How can an excitation have spin 1/2? It can, because of a beautiful mechanism known as fractionalization. Integrability will provide us with a rare example where we can understand neatly how fractionalization works with full analytical control.
4.1 Another electrostatic picture.

As we have seen, roughly speaking, complex Bethe roots describe ferromagnetic vacuum blocks. This leads to the intuition that anti-ferromagnetic physics should instead be related to purely real solutions. To study real solutions there is yet another electrostatic picture which is more convenient. Basically, the drawback of the picture in section 3.5 is that the forces represented in figure 14 are discontinuous which is sometimes inconvenient when dealing with real solutions. There is of course a simple fix to this, we simply write

\[ f(u) = f(u) + \pi - 2\pi \theta(u) - \pi + 2\pi \theta(u). \]  

(58)

where \( \tilde{f}(u) \) is now a continuous function, see figure 19. In fact \( \tilde{f}(u) = -2 \arctan(u) \). The key observation is that we can similarly define \( \tilde{F}(u) = L2 \arctan(2u) \) and cast Bethe equations using these new functions

\[ \tilde{F}(u_j) + \sum_{k \neq j}^{N} \tilde{f}(u_j - u_k) = 2\pi Q_j \]  

(59)

Note that compared to (49), here we denote the mode numbers with \( Q_j \). When going from untilded to tilded quantities we shifted out quantities by \( \pi \) and also by a step function times \( 2\pi \). The second we can actually ignore since in the exponent \( e^{x + 2\pi i \theta(x)} = e^x \). The shift by \( \pi \) simply means that depending on how many particles we have \( Q_j \) will differ from \( n_j \) by an integer or an half-integer. In short, the punch line is that \( Q_j \) is an integer if \( L + N - 1 \) is even while \( Q_j \) is an half-integer if \( L + N - 1 \) is odd. With this important caveat in mind, we can now simply use (59) where now all functions are smooth for real arguments.

This new electrostatic picture is very different from the previous one. For example, we see in (19) that the forces now vanish at coincident points. In particular, particles with the same mode number \( Q_j \) will therefore find the same equilibrium condition and thus have the same Bethe root \( u_j \). But this is not a good solution. After all, the Bethe ansatz wave function simply vanishes if two particles have identical rapidities because \( S(u, u) = -1 \). So
we conclude that we must always choose distinct mode numbers $Q_j$. Furthermore, note that we can not take $Q_j$ to be arbitrarily large since all the functions in the left hand side of (59). So there is a max value $Q_{\text{max}}$ that any mode number can take. We can estimate it by taking $u_j \to \infty$. This is not an obvious statement but we do not have time to do it properly so you will have to trust me here. In this way we get

$$Q_{\text{max}} = \frac{L}{2} - \frac{N}{2} - \frac{1}{2}$$

(60)

In sum: Selecting a solution to Bethe equations is equivalent to choosing a set of distinct $Q_j$ in the range $[-Q_{\text{max}}, +Q_{\text{max}}]$. The number of real solutions is the number of possible such choices.

4.2 Anti-ferromagnetic Vacuum

How many such choices do we have for the anti-ferromagnetic vacuum? Only one!, as we are now going to see. This is quite helpful to identify the antiferromagnetic state in the huge myriad of singlet states (it turns out that all other singlets have complex Bethe roots; there does not seem to be a clear physical argument about why this had to be the case a priori.).

For the anti-ferromagnetic vacuum we want a state of spin zero so we want to take the number of magnons – i.e. the number of spins up – to be half of the total number of spins so that $N = L/2$. But then the number of available mode numbers to pick from is

$$N = 2Q_{\text{max}} + 1 = L/2 = N$$

(61)

so that we have a single choice of picking the mode numbers

$$Q_j = \{-\frac{L}{4} + \frac{1}{2}, -\frac{L}{4} + \frac{3}{2}, \ldots, -\frac{L}{4} + \frac{3}{2}, -\frac{L}{4} - \frac{1}{2}\}$$

(62)

or

$$Q_j = -L/4 + 1/2 + (j - 1)$$

(63)

Now, in the large L limit all $u$’s will be very close together and we can define a smooth function $u(j) = u_j$ and its inverse $j(u)$. The derivative of the latter with respect to $u$ is a density since it counts how many roots there which is the definition of a density:

$$j(u) = \int_{-\infty}^{u} du' \frac{dj}{du'} \equiv \int_{-\infty}^{u} du' \rho(u')$$

(64)

So we can write (59) as

$$\tilde{F}(u(j)) + \int dk \tilde{f}(u(j) - u(k)) = 2\pi(-L/4 + 1/2 + (j - 1))$$

(65)

or

$$\tilde{F}(u) + \int dv \rho(v) \tilde{f}(u - v) = 2\pi(-L/4 + 1/2 + (j(u) - 1))$$

(66)

\(^7\)Note that there is no issue with $\sum_k$ vs $\sum_{k \neq j}$ since the summand vanishes for $k = j$ anyway.
which we can turn into a closed equation for the density $\rho$ by taking a derivative with respect to $u$,

$$\Phi(u) + \int dv \rho(v) \phi(u - v) = 2\pi \rho(u), \quad \Phi(u) = \tilde{F}'(u), \quad \phi(u) = \tilde{f}'(u). \quad (67)$$

Let us solve this equation using Fourier technology. Very explicitly, we multiply this equation by $e^{i\omega u}$ and integrate over $u$ to get

$$\hat{\Phi}(\omega) + \int dv \hat{\rho}(v)[\Phi(u)] \hat{\phi}(u - v) = 2\pi \hat{\rho}(\omega)$$

$$\Rightarrow \hat{\Phi}(\omega) + \frac{\hat{\rho}(\omega) \hat{\phi}(\omega)}{2\pi - \hat{\phi}(\omega)} = 2\pi \hat{\rho}(\omega)$$

Note that the analysis we just did is very general, valid for a generic S-matrix and dispersion provided the S-matrix is a function of the difference of rapidities. Let us then compute the Fourier transforms appearing in the right hand side for our Heisenberg chain. We have

$$\int du e^{i\omega u} \left[ \Phi(u) = \frac{L}{2i} \frac{d}{du} \log \frac{u - i/2}{u + i/2} = \frac{L}{i} \left( \frac{1}{u - i/2} - \frac{1}{u + i/2} \right) = \frac{L}{u^2 + 1/4} \right]. \quad (69)$$

Depending on the sign of of $\omega$ we can simply close the contour up or down by picking the residue at $u = \pm i/2$. In this way we get

$$\int du e^{i\omega u} \Phi(u) = 2\pi L e^{-|\omega|/2}, \quad (70)$$

and similarly

$$\int du e^{i\omega u} \phi(u) = -2\pi e^{-|\omega|}, \quad (71)$$

so that

$$\hat{\rho}(\omega) = \frac{L}{2\cosh(\omega/2)}. \quad (72)$$

As a check note that $\hat{\rho}(0) = \int du \rho(u) = L/2 = N$ as it should be. We actually do not need to compute $\rho(u)$ but we could of course do another Fourier transform to get it. Instead let us compute the energy

$$E/J = \sum \frac{1}{u_j^2 + 1/4} = \int du \rho(u) \frac{1}{u^2 + 1/4} = \int \frac{dw}{2\pi} \frac{\hat{\rho}(w)}{2\pi} e^{-|\omega|/2}$$

$$= 2L \int_0^\infty dw \frac{1}{e^w + 1} = 2L \int_0^1 \frac{dy}{y} \frac{1}{1/y + 1} = 2L \log(1 + y)^1 = 2L \log(2).$$

So the anti-ferromagnetic vacuum is a singlet with energy

$$E_{\text{anti-ferromagnetic vacuum}} = 2LJ \log(2). \quad (73)$$

Who to visualize this singlet? For the ferromagnetic vacuum we had a simple sea of spins $\downarrow$ along the chain, very simple to visualize. Is there some nice graphical depiction of the anti-ferromagnetic vacuum?

No. As far as I know it is a giant mess. We also do not know how to visualize the quantum chromodynamic vacuum of the real world. It is also, as far as we can tell, a big mess, albeit of course a very interesting one.
4.3 Holes and effective Bethe equations

It if now clear what we should do to study fluctuation close to the anti-ferromagnetic vacuum. We should simply repeat the above analysis for the case \( N = L/2 - 1 \). Something very simple but very important happens with \( Q_{\text{max}} \) when we take this \( N \). Namely, because we have one less interaction term in (59), \( Q_{\text{max}} \) decreases by one unit which means \( 2Q_{\text{max}} + 1 \) which is the number of elements in the lattice \([-Q_{\text{max}}, Q_{\text{max}}]\) increases by 2. In other words, we can now choose,

\[
Q = -Q_{\text{max}} + j - 1 + \theta(j - j_1) + \theta(j - j_2)
\] (74)

Said differently, instead of choosing which mode numbers in the lattice we pick we can choose two holes \( j_1, j_2 \), which are the mode numbers which we do not pick. So that

\[
dQ/du = \rho(u) + \delta(u - \theta_1) + \delta(u - \theta_2)
\] (75)

where \( \theta_1 = u(j_1) \) and \( \theta_2 = u(j_2) \). At this point, a long and interesting mathematical analysis akin to the one of the last section should be performed. Let me skip it and jump directly to the results: We find that the energy \( E/J = \sum_{j=1}^{N-1} \frac{1}{u_j + 1/4} \) is given by

\[
E/J = 2L \log(2) + \varepsilon(\theta_1) + \varepsilon(\theta_2), \quad \text{with} \quad e^{iLp(\theta_1)} S(\theta_1, \theta_2) = 1 = e^{iLp(\theta_2)} S(\theta_2, \theta_1),
\] (76)

where

\[
\varepsilon(\theta) = \frac{\pi}{2 \cosh \pi \theta}
\] (77)

\[
p(\theta) = \arctan(\sinh \pi \theta)
\] (78)

\[
S(\theta_1, \theta_2) = \frac{1}{i} \frac{\Gamma(\frac{1}{2} + i \frac{\theta_2}{2})(1 - i \frac{\theta_1}{2})}{\Gamma(\frac{1}{2} - i \frac{\theta_1}{2}) (1 + i \frac{\theta_2}{2})}, \quad \theta = \theta_1 - \theta_2.
\] (79)

Clearly, what this math is telling us is that the spin one change in the spin somehow broke apart into a pair of fractional particles of spin 1/2. This is fractionalization. Each particle has a dispersion relation \( \varepsilon(p) \) which can be easily worked out by solving this parametric representation and a these particles, called spinons, interact with a very non-trivial S-matrix. This is how we get to the results of figure 18.

4.4 AdS/CFT. String Theory and Integrability

Strings are 1+1d objects, see figure 20. Can we use integrability to learn about string theory – and therefore quantum field theory, – advance 21st century theoretical physics and improve our understanding of the physical universe?

I hope so. It definitely seems fun and worth trying! ²

²And I am obviously not alone. There is a huge community of hundreds of people making use of integrability techniques to learn about string theory and quantum field theory. You can have a look at Beisert et al review or the various IGST conferences for much more on this.

34
Figure 20: Right: String moving in time, sweeping a cylindrical worldsheet. Left: Quantum Spin Ring like the one studied in this lectures. Strings – like spin chains – are one dimensional objects. We can sometimes use integrability to learn about string theory dynamics!

4.5 Exercises

1. Work out the rapidity density \( \rho(u) \) whose Fourier transform is (72). You should find a somehow curious result: Up to a few constants, the Fourier transform is the function itself! What other function(s) do you know with this property? There is one which is very famous...

2. Show that (77) and (78) indeed lead to the dispersion in figure 18

3. Work out the math leading to (77), (78), (79).

4. Explicitly diagonalize the Heisenberg chain for a few small length like \( L = 2 \) and \( L = 3 \) (and perhaps a few larger ones using a computer), compute the anti-ferromagnetic energy and compare with the large \( L \) prediction derived in this lectures. You should see that the agreement is pretty reasonable, even for relatively small chains, and becomes quite good for larger chains.

5. Suppose we have a bunch of particles bouncing up and down in the presence of gravity. They only move vertically, along the \( y \) axis. When moving up or falling down they can meet other particles and scatter, see figure 21. How would you define integrability there? Can you imagine a nice solvable model in that case and solve it for a few excitations?\(^9\)

\(^9\)If yes, drop me an e-mail, I would be curious to have a look :)
Figure 21: If particles bounce in the presence of gravity, when they scatter they should do so as in flat space (equivalence principle). Is there some integrable version of this system with some nice pictures like 11 to go with? Note that here we have no three wave packets scattering at the same point but this could also happen of course. Are there special Integrable Bouncing Balls? See also exercise 5.