JUAN PABLO PAZ
Quantum Foundations and Information @ Buenos Aires
QUFIBA: http://www.qufiba.df.uba.ar
Departamento de Fisica Juan José Giambiagi, FCEyN, UBA, Argentina
Instituto de Fisica de Buenos Aires (Conicet UBA)

IFT SAIFR PERIMETER
SCHOOL
SAO PAULO
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Lecture 2: Kraus representation. Master equations. Examples. Application to Quantum Information. The effect of noise and dissipation on a qubit. How to protect a qubit. and an example.

Lecture 3: Quantum error correction: the basics. An

Lecture 4: Quantum Brownian Motion. Quantum to classical transition. Decoherence.
Albert Einstein (1954) in a letter to Max Born:

"Narrowness (localization) of quantum states with respect to macro - coordinates is not only independent of the principles of quantum mechanics but, moreover, it is incompatible with them."

• THE PROBLEM

- Set of possible quantum states is huge: (All states are allowed)
- Classical states: a (very!) small subset

• The solution: decoherence!!
DECOHERENCE PARADIGM

CLASSICALITY IS AN EMERGENT PROPERTY

A PREFERRED SET OF STABLE STATES IS DYNAMICALLY SELECTED

SYSTEM-ENVIRONMENT INTERACTION CREATES CORRELATIONS

DECOHERENCE: A RECORD OF THE STATE OF THE SYSTEM IS IMPRINTED IN THE ENVIRONMENT

TOO SIMPLE? ENOUGH TO UNDERSTAND CLASSICALITY
"DOUBLE SLIT" EXPERIMENT

\[ |\Psi(0)\rangle = \alpha |\varphi_1\rangle + \beta |\varphi_2\rangle \]

\[ |\Psi(t)\rangle = \alpha |\varphi_1(t)\rangle + \beta |\varphi_2(t)\rangle \]

\[ \text{Pr}(x,t) = |\alpha|^2 |\varphi_1(x,t)|^2 + |\beta|^2 |\varphi_2(x,t)|^2 + \]

\[ + 2 \text{Re}\left(\alpha\beta^* \varphi_1(x,t)\varphi_2^*(x,t)\right) \]

INTERFERENCE FRINGES
DECOHERENCE IN A DOUBLE SLIT EXPERIMENT

\[ \Psi(0) = (\alpha |\varphi_1\rangle + \beta |\varphi_2\rangle) \otimes |\varepsilon(0)\rangle \]

\[ \Psi(t) = \alpha |\varphi_1(t)\rangle \otimes |\varepsilon_1(t)\rangle + \beta |\varphi_2(t)\rangle \otimes |\varepsilon_2(t)\rangle \]

\[ \text{Pr}(x,t) = |\alpha|^2 |\varphi_1(x,t)|^2 + |\beta|^2 |\varphi_2(x,t)|^2 + 2 \text{Re} \left( \alpha \beta^* \varphi_1(x,t) \varphi_2^*(x,t) \langle \varepsilon_2(t) | \varepsilon_1(t) \rangle \right) \]

DECAY OF INTERFERENCE FRINGES
A MODEL: QUANTUM BROWNIAN MOTION (QBM)

QBM: Paradigm for a quantum open system (realistic in many cases: Caldeira-Leggett, etc)

System: Particle (harmonic oscillator)
Environment: Collection of harmonic oscillators
Interaction: bilinear

\[ H = H_S + H_E + H_{\text{int}} \]
\[ H_S = \frac{p^2}{2m} + V_0(x), \]
\[ H_E = \sum_n \left( \frac{p_n^2}{2m_n} + \frac{m_n \omega_n^2}{2} q_n^2 \right), \]
\[ H_{\text{int}} = \sum_n \lambda_n q_n x, \]

GOAL: Study evolution of the state of the system
‘State of the system’ : Reduced density matrix

\[ \rho_S = Tr_E(\rho_{SE}) \]
A MODEL: QUANTUM BROWNIAN MOTION (QBM)

Asumption (standard): Uncorrelated initial state

\[ t = 0 \quad \rho_{SE}(0) = \rho_S(0) \otimes \rho_E(0) \]

WILL SHOW THAT THE ONLY TWO RELEVANT “PARAMETERS” ARE
1) INITIAL STATE OF ENVIRONMENT (TEMPERATURE T),
2) SPECTRAL DENSITY OF ENVIRONMENT

\[ J(\omega) = \sum_n \frac{\chi_n^2}{2m_n \omega_n} \delta(\omega - \omega_n) \quad \rho_E(0) = \frac{e^{-\beta H_E}}{Z}, \beta = 1/k_B T \]

EXACT SOLUTION
EXACT MASTER EQUATION
DERIVATION OF THE MASTER EQUATION FOR QBM

EXACT EQUATION (FOR ALL TEMPERATURE AND SPECTRAL DENSITIES)

\[ \dot{\rho} = -i [H_R, \rho] - i \gamma(t) [x, \{p, \rho\}] - D(t) [x, [x, \rho]] - f(t) [x, [p, \rho]] \]

SIMPLIFIED DERIVATION (you should do this once in your lifetime!)

1) START FROM FULL SCHROEDINGER EQUATION

\[ \dot{\rho}_T = - i [H_T, \rho_T] \]

2) GO TO THE INTERACTION PICTURE AND USE DYSON EXPANSION

\[ H_0 = H_S + H_E; \quad U_0 = \exp(-iH_0 t); \quad \tilde{\rho}_T = U_0^{-1} \rho_T U_0; \quad \tilde{H}_{\text{int}} = U_0^{-1} H_{\text{int}} U_0 \]

\[ \tilde{\rho}_T = - i [\tilde{H}_{\text{int}}, \tilde{\rho}_T] \Rightarrow \]

\[ \tilde{\rho}_T (t) = \sum_{n=0}^{\infty} \int_{t_0}^{t} dt_1 \cdots \int_{t_0}^{t_{n-1}} dt_n \frac{1}{i^n} \left[ \tilde{H}_{\text{int}} (t_1), \ldots, \left[ \tilde{H}_{\text{int}} (t_n), \tilde{\rho}_T (t_0) \right] \right] \]
DERIVATION OF THE MASTER EQUATION FOR QBM

3) USE SECOND ORDER PERTURBATION THEORY

\[
\tilde{\rho}_T(t) = \tilde{\rho}_T(t_0) + \frac{1}{i} \int_{t_0}^{t} dt_1 \left[ \tilde{H}_{\text{int}}(t_n), \tilde{\rho}_T(t_0) \right] \\
- \int_{t_0}^{t} dt_1 \int_{t_0}^{t_2} dt_2 \left[ \tilde{H}_{\text{int}}(t_1), \left[ \tilde{H}_{\text{int}}(t_2), \tilde{\rho}_T(t_0) \right] \right]
\]

4) TAKE TIME DERIVATIVE AND TRACE OVER THE ENVIRONMENT

\[
\dot{\tilde{\rho}}_T(t) = \frac{1}{i} \left[ \tilde{H}_{\text{int}}(t), \tilde{\rho}_T(t_0) \right] - \int_{t_0}^{t} dt_1 \left[ \tilde{H}_{\text{int}}(t), \left[ \tilde{H}_{\text{int}}(t_1), \tilde{\rho}_T(t_0) \right] \right]
\]

\[
\tilde{\rho} = Tr_E \left( \tilde{\rho}_T \right)
\]

\[
\dot{\tilde{\rho}}(t) = \frac{1}{i} Tr_E \left[ \tilde{H}_{\text{int}}(t), \tilde{\rho}_T(t_0) \right] - \int_{t_0}^{t} dt_1 Tr_E \left[ \tilde{H}_{\text{int}}(t), \left[ \tilde{H}_{\text{int}}(t_1), \tilde{\rho}_T(t_0) \right] \right]
\]
5) ASSUME SEPARABLE INITIAL STATE

\[ \dot{\rho}(t) = \frac{1}{i} Tr_E \left[ \tilde{H}_{\text{int}}(t), \tilde{\rho}(t_0) \otimes \rho_E \right] - \int_{t_0}^{t} dt_1 Tr_E \left[ \tilde{H}_{\text{int}}(t_1), \tilde{\rho}(t_0) \otimes \rho_E \right] \]

6) TRICK: REPLACE INITIAL STATE OF SYSTEM IN R.H.S. OF EQUATION!

\[ \tilde{\rho}(t) = \tilde{\rho}(t_0) + \frac{1}{i} \int_{t_0}^{t} dt Tr_E \left[ \tilde{H}_{\text{int}}(t_1), \tilde{\rho}(t_0) \otimes \rho_E \right] \]

7) UP TO SECOND ORDER THE EQUATION IS LOCAL IN TIME!

\[ \dot{\rho}(t) = \frac{1}{i} Tr_E \left[ \tilde{H}_{\text{int}}(t), \tilde{\rho}(t) \otimes \rho_E \right] - \int_{0}^{t} dt_1 Tr_E \left[ \tilde{H}_{\text{int}}(t), \left[ \tilde{H}_{\text{int}}(t_1), \tilde{\rho}(t) \otimes \rho_E \right] \right] \]

\[ + \int_{0}^{t} dt_1 Tr_E \left[ \tilde{H}_{\text{int}}(t), Tr_E \left( \left[ \tilde{H}_{\text{int}}(t_1), \tilde{\rho}(t) \otimes \rho_E \right] \right) \otimes \rho_E \right] \]
DERIVATION OF THE MASTER EQUATION FOR QBM

8) WRITE INTERACTION TERM, GO BACK TO SCHRODINGER PICTURE, AND REORDER TERMS (DO IT!)

\[ H_{\text{int}} = \sum_k S_k \otimes E_k \quad \rho = U_S \tilde{\rho} U_S^{-1}, \quad U_S = e^{-i H_S t} \]

\[ F_k(t) = Tr_E \left( \rho_E E_k(t) \right) \]

\[ \dot{\rho} = -i [H_S, \rho] - i \sum_k [S_k, \rho] F_k(t) \]

\[ + \frac{i}{2} \sum_{kk'} \int_0^t dt_1 \eta_{kk'}(t, t_1) \left[ S_k, \{ S_{k'}(t_1 - t), \rho \} \right] \]

\[ - \frac{1}{2} \sum_{kk'} \int_0^t dt_1 \nu_{kk'}(t, t_1) \left[ S_k, [S_{k'}(t_1 - t), \rho] \right] \]

\[ \eta_{kk'}(t, t') = \frac{i}{2} Tr_E \left( \rho_E \left[ E_k(t), E_{k'}(t') \right] \right) \]

\[ \nu_{kk'}(t, t') = \frac{1}{2} Tr_E \left( \rho_E \left\{ E_k(t), E_{k'}(t') \right\} \right) - F_k(t) F_{k'}(t') \]
DERIVATION OF THE MASTER EQUATION FOR QBM

9) NOW CONSIDER QBM

\( H_{\text{int}} = x \otimes \sum_k \lambda_k q_k \)

\( \rho_E = \exp\left(-H_E / k_B T\right) / Z \)

\( E_k(t) \rightarrow q_k(t) = q_k \cos(\omega_k t) + \frac{1}{m_k \omega_k} p_k \sin(\omega_k t) \)

\( F_k(t) = \left\langle \lambda_k q_k(t) \right\rangle = 0 \)

\( \eta_{kk'}(t,t') = \delta_{kk'} \frac{\lambda_k^2}{2m_k \omega_k} \sin(\omega_k (t-t')) \)

\( \nu_{kk'}(t,t') = \delta_{kk'} \frac{\lambda_k^2}{2m_k \omega_k} \cos(\omega_k (t-t')) (1 + 2n_k) \)

\( J(\omega) = \sum_k \frac{\lambda_k^2}{2m_k \omega_k} \delta(\omega - \omega_k) \)

\( \nu(t) = \sum_k \frac{\lambda_k^2}{2m_k \omega_k} \cos(\omega_k (t-t')) (1 + 2n_k) \)

\( \eta(t) = \sum_k \frac{\lambda_k^2}{2m_k \omega_k} \sin(\omega_k (t-t')) \)

\( n_k = \left(e^{\omega_k / k_B T} - 1\right)^{-1} \)

\( \eta(t) = \int_0^\infty d\omega \sin(\omega t) J(\omega) \)
DERIVATION OF THE MASTER EQUATION FOR QBM

10) REWRITE MASTER EQUATION ALMOST IN FINAL FORM

\[ \dot{\rho} = -i[H_S, \rho] - \frac{1}{2} \int_0^t dt_1 \left( -i \eta(t_1) \left[ x, \{ x(-t_1), \rho \} \right] + \nu(t_1) \left[ x, \{ x(-t_1), \rho \} \right] \right) \]

11) REPLACE \( x(t) \) IN THE R.H.S. OF THE MASTER EQUATION

\[ x(t) = x \cos(\Omega t) + \frac{1}{m\Omega} p \sin(\Omega t) \]

\[ \dot{\rho} = -i[H_S, \rho] - i \frac{m}{2} \delta \omega^2(t) \left[ x, \{ x, \rho \} \right] - i \gamma(t) \left[ x, \{ p, \rho \} \right] - D(t) \left[ x, \{ x, \rho \} \right] - f(t) \left[ x, \{ p, \rho \} \right] \]

\[ \delta \omega^2(t) \approx -\frac{2}{m} \int_0^t dt' \cos(\Omega t') \eta(t') \]

\[ \gamma(t) \approx \frac{1}{m\Omega} \int_0^t dt' \sin(\Omega t') \eta(t') \]

\[ D(t) \approx \frac{1}{2} \int_0^t dt' \cos(\Omega t') \nu(t') \]

\[ f(t) \approx \frac{1}{2m\Omega} \int_0^t dt' \sin(\Omega t') \nu(t') \]
WE DERIVED THE MASTER EQUATION FOR QBM!

\[
\dot{\rho} = -i \left[ H_S + \frac{m}{2} \delta \omega^2(t)x^2, \rho \right] - i\gamma(t) [x, \{p, \rho\}] - D(t) [x, [x, \rho]] - f(t) [x, [p, \rho]]
\]

Time dependent coefficients are determined by spectral density and initial temperature

**COMMENTS:**

1) EQUATION WAS DERIVED TO SECOND ORDER IN THE INTERACTION BUT IT IS AN EXACT EQUATION VALID TO ALL ORDERS.


2) IN SOME CASES ANALYTIC EXPRESSIONS FOR THE TIME DEPENDENT COEFFICIENTS CAN BE OBTAINED
\[ \dot{\rho} = -i \left[ H_s + \frac{m}{2} \delta \omega^2(t)x^2, \rho \right] - i\gamma(t)[x, \{p, \rho\}] - D(t)[x, [x, \rho]] - f(t)[x, [p, \rho]] \]

**RENORMALIZATION AND DAMPING**

\[ \langle \dot{p} \rangle = -m\Omega^2_R(t)\langle x \rangle - 2\gamma(t)\langle p \rangle \]
\[ \langle \dot{x} \rangle = \langle p \rangle / m \]
\[ \Omega^2_R(t) = \Omega^2 + \delta \omega^2(t) \]

- Normal friction (constant \( \gamma(t) \)): ohmic environment

\[ J(\omega) = 2m\gamma \frac{\omega}{\pi} \frac{\Lambda^2}{\Lambda^2 + \omega^2} ; \quad \gamma(t) = \gamma \frac{\Lambda^2}{\Lambda^2 + \omega^2} \left( 1 - \left( \cos(\Omega t) + \frac{\Lambda}{\Omega} \sin(\Omega t) \right) e^{-\Lambda t} \right) \]
\[ \begin{align*}
\dot{\rho} &= -i \left[ H_s + \frac{m}{2} \delta \omega^2(t)x^2, \rho \right] - i\gamma(t) [x, \{p, \rho\}] - D(t) [x, [x, \rho]] - f(t) [x, \{p, \rho\}] \\
\frac{d\langle p^2 \rangle}{dt} &= -m\Omega_R^2(t)\langle xp + px \rangle - 4\gamma(t)\langle p^2 \rangle + 2D(t) \\
\frac{d\langle xp + px \rangle}{dt} &= 2\frac{\langle p^2 \rangle}{m} - 2m\Omega_R^2(t)\langle x^2 \rangle - 2\gamma(t)\langle xp + px \rangle - 2f(t) \\
\frac{d\langle x^2 \rangle}{dt} &= \frac{1}{m}\langle xp + px \rangle \\
\end{align*} \]

- Diffusion coefficients (D(t) and f(t)) depend on spectral density and temperature

\[ 
D(t) \approx \int_0^t dt' \cos(\Omega t')\nu(t') \quad f(t) \approx -\frac{1}{M\Omega} \int_0^t dt' \sin(\Omega t')\nu(t') \quad \nu(t) = \int_0^\infty d\omega \cos(\omega t) \coth\left(\frac{\omega}{kT}\right)J(\omega) 
\]

\[ 
\langle p^2 \rangle \rightarrow \frac{D}{2\gamma} \\
\frac{m\Omega_R^2}{2} \langle x^2 \rangle \rightarrow \frac{D}{2m\gamma} - f 
\]
BEHAVIOR OF COEFFICIENTS

Ohmic environment

$$J(\omega) = 2m\gamma \frac{\omega}{\pi} \frac{\Lambda^2}{\Lambda^2 + \omega^2} \rightarrow 2m\gamma \frac{\omega}{\pi} ; \quad \gamma(t) \rightarrow \gamma \quad (t >> \Lambda^{-1})$$

- Diffusion coefficients (D(t) and f(t)) have initial transient and approach temperature-dependent asymptotic values

$$\langle p^2 \rangle \rightarrow \frac{D}{2\gamma}$$

$$m\Omega_R^2 \langle x^2 \rangle \rightarrow \frac{D}{2m\gamma} - f$$
Use this to investigate:

1) What is the decoherence timescale?,

2) What are the pointer states?

Ohmic environment in a high temperature initial state

\[ J(\omega) = 2m\gamma \omega \quad (\omega \leq \Gamma), \]
\[ k_B T \gg \hbar \Omega \]

\[ \gamma(t) \to \gamma, \quad D(t) \to 2m\gamma k_B T, \quad f(t) \to 0 \]

Approximate master equation (ohmic, high temperature)

\[ \dot{\rho} = -i[H_R, \rho] - i\gamma \left[ x, \{ p, \rho \} \right] - D \left[ x, [x, \rho] \right] \]
DECOHERENCE IN QUANTUM BROWNIAN MOTION

CLASSICAL PHASE SPACE

A superposition of two quantum states (the particle is here AND there: quantum interference)

THE WIGNER REPRESENTATION OF A SCHROEDINGER CAT STATE (QUASI-PROBABILITY DISTRIBUTION)
DECOHERENCE IN QUANTUM BROWNIAN MOTION: MAIN RESULTS ARE BETTER SEEN
REPRESENTING THE STATE IN PHASE SPACE VIA WIGNER FUNCTIONS

\[ W(x, p) = \int \frac{dy}{2\pi\hbar} e^{ipy/\hbar} \langle x - y/2|\rho|x + y/2\rangle \]

**PROPERTIES:**
- \( W(x, p) \) is real
- Use it to compute inner products as:
  \[ \int dx dp \, W_1(x, p) W_2(x, p) = \frac{1}{2\pi\hbar} Tr(\rho_1 \rho_2) \]
- Integral along lines give all marginal distributions:
  \[ \int dx dp \, W(x, p) = \text{Probability}(ax + bp = c) \]

MASTER EQUATION CAN BE REWRITTEN FOR THE WIGNER FUNCTION: IT HAS THE FORM OF A
FOKER-PLANCK EQUATION

\[ \dot{W} = \{H_0, W\}_\text{MB} + \gamma \partial_p (p W) + D \partial_{pp}^2 W + f \partial_{xp}^2 W \]
DECOHERENCE TIMESCALE
(JPP & W.H. Zurek)

\[ QBM \Rightarrow \Gamma_{DECO} = \gamma_{RELAXATION} \left( \frac{L}{\lambda_{DB}} \right)^2, \]

\[ \lambda_{DB} = \frac{\hbar}{\sqrt{2 m k_B T}} \]

\[ m = 1gr, \quad T = 300K, \quad L = 1cm \]

\[ \Rightarrow \Gamma_{DECO} = 10^{40} \gamma_{RELAXATION}, \]

REVERSIBLE CLASSICAL LIMIT EXISTS
Serge Haroche: breeding Schroedinger’s cat in a cavity and observing decoherence

Measured Wigner function! (probe the cat sending Schroedinger mice!)

Haroche, et al, 2006 Nature
50ms in the life of a cat…
(from “dead AND alive” to “dead OR alive”)
What states are selected?

Not so simple

Operational definition of pointer states

“Predictability sieve”

1981: determined by interaction Hamiltonian

\[ |\Psi(0)\rangle\langle\Psi(0)| \quad t \quad \rho(t) \]

Initial state of the system (pure) \quad State of system at time t (mixed)

Measure degradation induced by the environment (purity loss, entropy increase)

S_{\text{rho}}(t) = Tr(\rho(t) \ln \rho(t)) = \ln \rho(t) = Tr(\rho^2(t))

Predictability sieve: find the initial states such that these quantities are minimized (for a dynamical range of times)
A SIMPLE SOLUTION FROM THE PREDICTABILITY SIEVE CRITERION FOR THE QUANTUM BROWNIAN MOTION

\[ \text{Tr} \rho^2 (t) - \text{Tr} \rho^2 (0) = -2 D \Delta x^2 + m^2 \omega^2 \Delta p^2 + \ldots \]

Solve the problem. Compute change in purity.

Simple approximate expression (small damping, state almost pure, etc)

POINTER STATES ARE COHERENT STATES!!

"position eigenstate"

"momentum eigenstate"

Coherent States via Decoherence

Wojciech H. Zurek, (a) Salman Habib, (b) and Juan Pablo Paz (c)

Theoretical Astrophysics, Mail Stop B288, Los Alamos National Laboratory, Los Alamos, New Mexico 87545
(Received 10 June 1992)
1) Quantum Measurement: System’s evolution is negligible. Pointer basis is eigenbasis of the interaction Hamiltonian (Zurek 81)

2) Dynamical case (QBM): Pointer basis arises from competition between system and environment

3) Lazy environment regime: Environment is “slow”: Pointer states are eigenstates of the Hamiltonian of the system! JPP. & W.Zurek, PRL 82, 5181 (1999)

TAILOR MADE POINTER STATES ARE POSSIBLE?

YES Environmental engeneering…
DECOHERENCE IS AN ESSENTIAL INGREDIENT TO UNDERSTAND THE TRANSITION FROM QUANTUM TO CLASSICAL

UNDERSTANDING THE PROCESS OF DECOHERENCE REQUIRES UNDERSTANDING THE PHYSICS OF QUANTM OPERN SYSTEMS